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Sard's Theorem

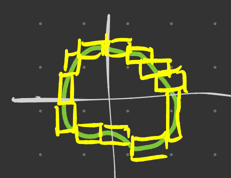
$f : M \longrightarrow N$ smooth then $\underbrace{\mu}_{\text{measure}}(f(\underbrace{\mathrm{crit}(f)}_{}))=0$

critical pts of f, i.e. $p \in M$ s.t. $df_p : T_pM \longrightarrow T_pN$ **not** surjective

Arthur Sard
1909 – 1980

$A \subseteq \mathbb{R}^n$ has measure 0 when $\forall \delta > 0$ $\exists$ countable collection of rectangles $R_i \subseteq \mathbb{R}^n$ covering A & with $\sum_i \underbrace{\mathrm{vol}(R_i)}_{\prod_k (b_k - a_k)} < \delta$.

$[a_1,b_1] \times \cdots \times [a_n,b_n]$



(presently!)

We won't develop a general notion of measure for $A \subseteq M$ smooth mfld but we will define measure 0 for $A \subseteq M$ and write $\mu(A) = 0$.

Prop Suppose $A \subseteq \mathbb{R}^n$ has $\mu(A) = 0$ and $F : A \to \mathbb{R}^n$ is smooth. Then $\mu(F(A)) = 0$.

Pf Choose a cover of A by countably many precompact open $U \subseteq \mathbb{R}^n$ on which F may be smoothly extended. Then $F(A)$ is a union of countably many sets of the form $F(A \cap \bar{U})$ and it suffices to show each of these has $\mu = 0$.

Since $\bar{U}$ compact, $\exists C \in \mathbb{R}$ s.t. $|Fx - Fx'| \leq C|x - x'|$ $\forall x, x' \in U$.

Given $\delta > 0$, choose countable cover $\{B_j\}$ of $A \cap \bar{U}$ with

$\sum \text{vol}(B_j) < \delta$. By A, $F(\bar{U} \cap B_j) \subseteq \tilde{B}_j$.

ball of radius $\leq C \cdot \text{radius}(B_j)$

Thus $F(A \cap \bar{U}) \subseteq \bigcup \tilde{B}_j$ and sum of volumes of RHS is $\leq C^n \delta$. Given $\delta' > 0$, take $\delta = \frac{\delta'}{2C^n}$ to bound vol of cover by δ. $\square$

Justifies defining $A \subseteq M$ has measure 0 when for every smooth chart (U, φ) on M, $\varphi(A \cap U) \subseteq \mathbb{R}^n$ has measure 0.

Equivalently (p. 128) $\exists$ collection of smooth charts $\{(U_\alpha, \varphi_\alpha)\}$ covering A with $\mu(\varphi(A \cap U_\alpha)) = 0 \ \forall \alpha$.

Facts

- $A \subseteq M$ measure $0 \Rightarrow M \setminus A \subseteq M$ dense
- $F: M \to N$ smooth, $A \subseteq M$ measure $0 \Rightarrow F(A) \subseteq N$ measure 0

Sard's Thm $\underset{\text{dim } m}{F: M} \longrightarrow \underset{n}{N}$ smooth, then $\mu(F(\text{crit } F)) = 0$

Pf by induction on m.

Base case $m = 0$ ✓

Now suppose $m \geq 1$ and Sard's thm holds for all domains of smaller dimn. By covering M, N with countably many smooth charts may assume $F: U \longrightarrow \mathbb{R}^n$ smooth.

$$\underset{\substack{x^1, \dots, x^m}}{\underset{\mathbb{R}^m}{\text{open } U \subseteq}} \qquad\qquad y^1, \dots, y^n$$

Let $C = \mathrm{crit}(F) \subseteq U$. Filter $C \supseteq C_1 \supseteq C_2 \supseteq \cdots$ with

$$C_k = \{x \in C \mid i\text{th order partial deriv's of } F \text{ vanish at } x \text{ for } 1 \le i \le k\}.$$

By continuity, all C_k are closed in U. WTS $\mu(F(C)) = 0$.

Do so in 3 steps:

Step 1: $\mu(F(C \setminus C_1)) = 0$

Step 2: $\mu(F(C_k \setminus C_{k+1})) = 0 \quad \forall k$

don't handle the case of x at which all paritial derivs vanish

Step 3: For $k > \frac{m}{n} - 1$, $\mu(F(C_k)) = 0$. finishes the proof

Step 1: $\mu(F(C \setminus C_1)) = 0$: C_1 closed so may replace U with $U \setminus C_1$ and assume $C_1 = \emptyset$. For $a \in C$, reorder coords so that $\frac{\partial F^1}{\partial x^1}(a) \neq 0$.

In a nbhd V_a of a take coords $u = F^1, v^2 = x^2, \dots, v^m = x^m$.

Shrink so $\bar{V}_a$ compact and coords extend smoothly to $\bar{V}_a$.

In these coords, F has rep'n $\underbrace{(u, F^2(u,v), \dots, F^n(u,v))}_{}$ with

?

$$\begin{tikzcd} U \arrow[r, "F = (F^1, \dots, F^n)"] \arrow[d, "(F^1, x^2, \dots, x^m)"', "\wr"] & \mathbb{R}^n \\ V \arrow[ur] & \end{tikzcd}$$

Use chain rule!

$$JF(u,v) = \begin{pmatrix} 1 & 0 \\ * & \boxed{\dfrac{\partial F^i}{\partial v^j}} \end{pmatrix}$$

Thus $C \cap \bar{V}_a$ is the locus where $\boxed{\frac{\partial F^i}{\partial v^j}}$ has rank $< n-1$.

WTS $F(C \cap \bar{V}_a)$ has measure 0. $C \cap \bar{V}_a$ is compact, so may check that each slice $F(C \cap \bar{V}_a) \cap \{y^1 = c\}$ has $(n-1)$-dim'l measure 0 (Fubini).

Set $F_c(v) = (F^2(c,v), \dots, F^n(c,v))$. By ind'n hypothesis, crit ~~pts~~ values of F_c have measure 0 and these are exactly the crit ~~pts~~ vals of F in $C \cap \bar{V}_a$ with $F^1 = c$. ✓

<u>Step 2</u>: $\mu(F(C_k \setminus C_{k+1})) = 0 \quad \forall k$

Take $a \in C_k \setminus C_{k+1}$ and let $y : U \to \mathbb{R}$ be a k-th order partial deriv of F with a nonvanishing $\frac{\partial y}{\partial x^i}(a)$. Then a is a regular point of y so $\exists$ nbhd V_a of reg pts of y. Set $Y = \{x \mid y(x) = 0\} \subseteq V_a$; its a regular hypersurface. Have $C_k \cap V_a \subseteq Y$ by dfn of C_k.

For $p \in C_k \cap V_a$, dF_p is not surj, so neither is $d(F|_Y)_p = (dF_p)|_{T_pY}$

Thus $F(C_k \cap V_a) \subseteq F(\underbrace{\mathrm{crit}(F|_Y)}_{\text{measure 0 by ind hypothesis}})$.

Now $C_k \setminus C_{k+1}$ is covered by countably many such V_a ✓

Step 3: For $k > \frac{m}{n} - 1$, $\mu(F(C_k)) = 0$

U can be covered w/ countably many closed cubes $E \subseteq U$

so suffices to show $\mu(F(C_k \cap E)) = 0$ $\forall$ such E.

Take A bounding $| \, |$ of $(k+1)$st order ∂ derivs of F in E

let R be side length of E and $K \in \mathbb{Z}_+$ chosen later.

Subdivide E into K^m cubes of side length R/K, $E_1, \dots, E_{K^m}$.

For $a_i \in E_i \cap C_k$, $\exists A' = A'(A, k, m)$ s.t.

$$|F(x) - F(a_i)| \leq A' |x - a_i|^{k+1} \qquad (\text{Taylor})$$

So $F(E_i) \subseteq$ ball radius $A'(R/K)^{k+1}$. Thus $F(C_k \cap E) \subseteq$

union of K^m balls, sum of vols $\leq$

$$K^m A'^n (R/k)^{n(k+1)} = A'' K^{m-nk-n}$$

$$A'^n R^{n(k+1)}$$

Since $k > m/n - 1$, exponent of K is $< m - n\left(\frac{m}{n} - 1\right) - n$

$$= m - m + n - n$$

$$= 0 \qquad (\text{or } n = 0 \Rightarrow C = \emptyset)$$

So taking $k \gg 0$ get vol arbitrarily small. ✓ $\square$

Cor $F: M \longrightarrow N$ smooth, $\dim M < \dim N$ then $\mu(F(M)) = 0$. $\square$

Whitney embedding: $\dim M = n$ then $\exists$ emb

$$M \hookrightarrow \mathbb{R}^{2n+1}$$