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Immersed Submanifolds

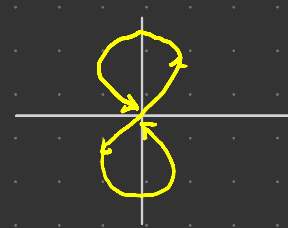
M smooth mfld w/ or w/o ∂

not necessarily subspace topology

$S \subseteq M$ endowed with a topology wrt which S is a top'l mfld and a smooth structure s.t. $S \hookrightarrow M$ is a smooth immersion is an immersed submanifold of M.

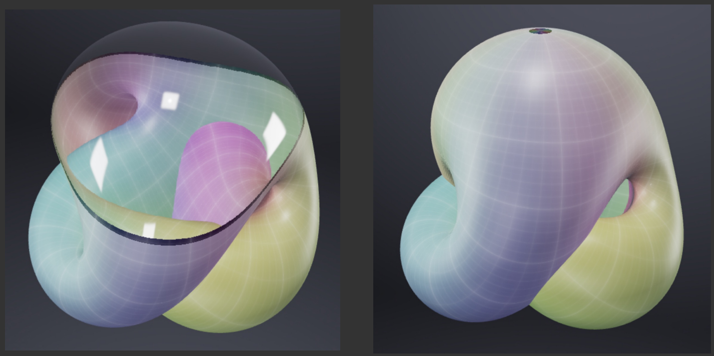
E.g.
- embedded submflds
- figure-eight $\beta : (-\pi, \pi) \longrightarrow \mathbb{R}^2$ immersion's image

$$t \longmapsto (\sin 2t, \sin t)$$



- image of any injective smooth immersion

  ↳ many texts allow self-crossings



Boy's surface : smooth immersion $\mathbb{RP}^2 \longrightarrow \mathbb{R}^3$; not an immersed smooth manifold

- Image of dense curve in torus $\gamma : \mathbb{R} \longrightarrow T^2$

  $t \longmapsto (e^{2\pi i t}, e^{2\pi i \alpha t})$

  irrational

Immersed submflds are locally embedded b/c immersions are local embeddings.

means immersed submflds henceforth

Restriction to submflds

Thm $F : M \to N$ smooth, $S \subseteq M$ submfld, then $F|_S : S \to N$ is smooth

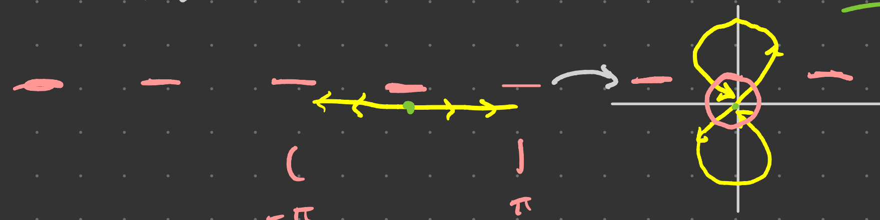
Pf $F|_S = F \circ \iota$ for $\underbrace{\iota : S \hookrightarrow M}_{\text{smooth by defn of immersed}}$. $\square$

Thm $S \subseteq M$ submfld, $F : N \to M$ smooth with $F(N) \subseteq S$. Then $F : N \to S$ is smooth iff it is cts.

Huh?! How could $F: N \to S$ not be cts??

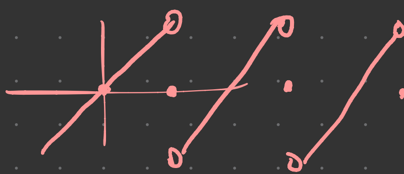
- If $S \subseteq M$ is embedded, then S has subspace top so $F: N \to S$ is automatically cts.

- For β the immersed figure eight, $S \subseteq \mathbb{R}^2$ consider $G: \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (\sin(2t), \sin t)$ which is smooth. But $G: \mathbb{R} \to S$ is not cts!



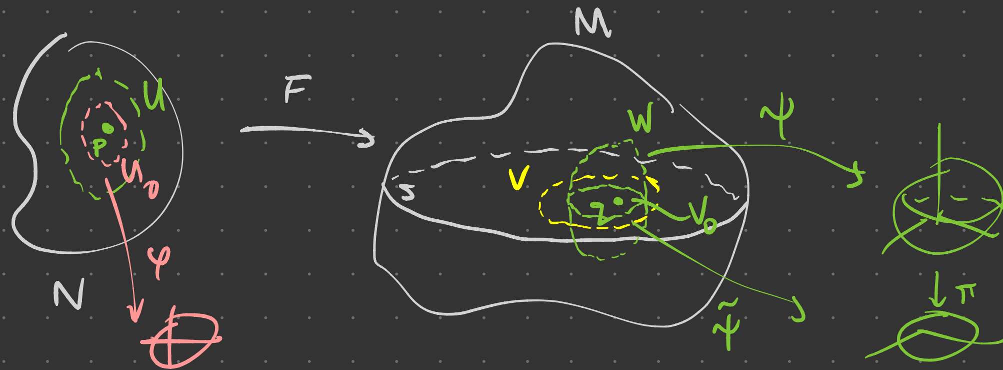
$\beta^{-1} \circ G$ not cts.

$\mathbb{R} \to (-\pi, \pi)$



Pf of Thm Smooth $\Rightarrow$ cts. ✓

Suppose $F: N \to S$ cts.



- $F(p) = q \in S$. Take V nbhd of q in S s.t. $\iota|_V : V \hookrightarrow M$ is a sm emb.

  Take (W, ψ) smooth slice chart

  Since $V_0 = (\iota|_V)^{-1}(W)$ open in V, it's also open in S, so $(V_0, \tilde{\psi})$ is a smooth chart for S.

- Take $U = F^{-1} V_0 \subseteq N$ open containing p (via continuity).

  Take (U_0, φ) sm chart for N w/ $p \in U_0 \subseteq U$.

Then coord rep'n of $F: N \to S$ wrt (U_0, φ), $(V_0, \tilde{\psi})$ is

$$\tilde{\psi} \circ F \circ \varphi^{-1} = \pi \circ \underbrace{(\psi \circ F \circ \varphi^{-1})}$$

harn as $F: N \to M$

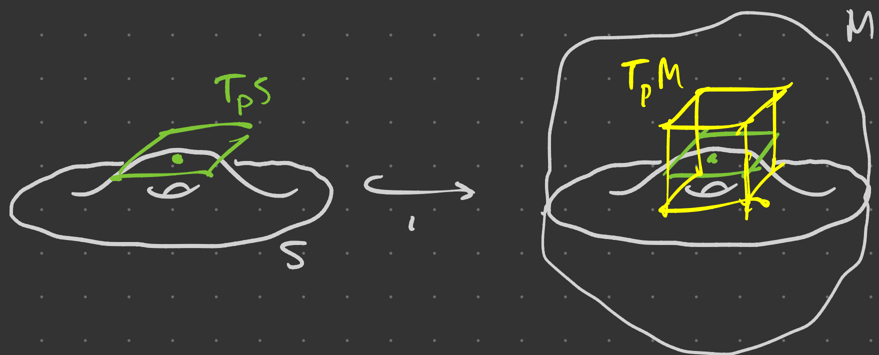
both smooth!

Thus $F: N \to S$ is smooth. $\square$

Uniqueness of smooth structures

Thm $S \subseteq M$ emb submfld. The subspace top on S and smooth structure as before are the only topology + sm str wrt which S is an embedded or immersed mfld. **pf** p. 114 $\square$

Tangent spaces to submflds



$\iota : S \hookrightarrow M$ induces $d\iota_p : T_pS \longrightarrow T_pM$

$$v \longmapsto \tilde{v}$$

Here $\tilde{v} f = d\iota_p(v) f = v(f \circ \iota) = v(f|_S)$ for $f \in C^\infty(M)$

Use $d\iota_p$ to identify T_pS w/ its image in T_pM.

Prop $S \subseteq M$ submfld, $p \in S$. A vector $v \in T_p M$ is in $T_p S$ iff $\exists$ smooth curve $\gamma : J \to M$ with image $\subseteq S$ s.t. $\gamma : J \to S$ smooth, $0 \in J$, $\gamma(0) = p$, and $\gamma'(0) = v$.

Prop Also for $S \subseteq M$ embedded submfld

$$T_p S = \{ v \in T_p M \mid vf = 0 \text{ whenever } f \in C^\infty(M) \text{ and } f|_S = 0 \}$$

Directional derivatives that are 0 for fns constant on S.

Pf Suppose $v \in T_p S \subseteq T_p M$. Then $v = d\iota_p(w)$ for some $w \in T_p S$, $\iota : S \hookrightarrow M$. If $f|_S = 0$, then $vf = w(\underbrace{f \circ \iota}_{f|_S}) = 0$. ✓

coordinate conditions for $v \in T_pS$

Now suppose $v \in T_pM$ and $vf = 0$ for $f|_S = 0$.

Let $x^1, \dots, x^n$ be slice coords for S in some nbhd U of p,

so $U \cap S = \{x^{k+1} = \dots = x^n = 0\}$ and $x^1, \dots, x^k$ are coords for $U \cap S$.

$\iota : U \cap S \hookrightarrow M$ has coord rep $(x^1, \dots, x^k) \longmapsto (x^1, \dots, x^k, 0, \dots, 0)$.

Thus $T_pS = \mathrm{span}\left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^k}\Big|_p \right\}$. Write

$$v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\Big|_p .$$

Then $v \in T_pS$ iff $v^{k+1} = \dots = v^n = 0$.

Let φ be a smooth bump fn supported in U and equal to 1 in a nbhd of p. Choose index $j > k$, set

$$f(x) = \varphi(x) x^j$$

extended to be 0 on $M \setminus \operatorname{supp} \varphi$. Then $f|_S = 0$

$$\Rightarrow \quad 0 = vf = \sum v^i \frac{\partial f}{\partial x^i}(p) = v^j.$$

Thus $v \in T_pS$ as desired. $\square$

Prop $S \subseteq M$ embedded submfld and $\Phi: U \to N$ local defining map for S (i.e. $U \cap S = \Phi^{-1}\{c\}$ reg level set). Then

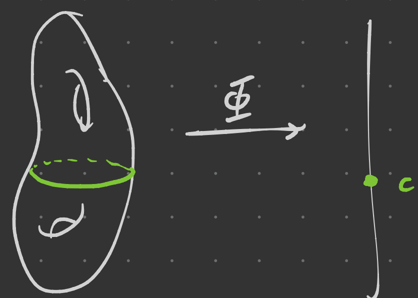
$$T_pS = \ker d\Phi_p : T_pM \to T_pN.$$

Pf $\Phi \circ \iota$ is constant on $S \cap U$, so $d\Phi_p \circ d\iota_p = 0 : T_pS \to T_{\Phi(p)}N$.

Thus $T_pS \subseteq \ker d\Phi_p$. Now $d\Phi_p$ is surjective so by rank-nullity

$$\dim \ker d\Phi_p = \dim T_pM - \dim T_{\Phi(p)}N = \dim T_pS.$$

Thus $T_pS = \ker d\Phi_p$. $\square$



Cor $S \subseteq M$ level set of a smooth submersion $\Phi = (\Phi^1, \dots, \Phi^k) : M \to \mathbb{R}^k$ then $v \in T_pM$ is in T_pS iff $v\Phi_1 = \dots = v\Phi_k = 0$.

$\square$

$\underline{Q}$ Given $\iota : S \hookrightarrow M$ immersion

can we produce local defining fn Φ for ι?

$\underline{A}$ Not just from $d\iota$ info, but

yes from slice charts.

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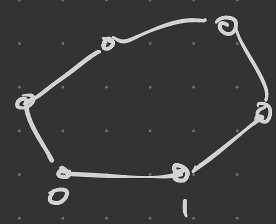
space of equilateral n-gons

$$M_n = \{ (z_1, \dots, z_{n-1}) \in \mathbb{T}^{n-1} \mid 1 + z_1 + \cdots + z_n = 0 \}$$



$\underline{Q}$ Is $M_n \subseteq \mathbb{T}^{n-1}$ with subspace top an emb or immersed submfld ?

$M_n = \sigma^{-1}\{-1\}$ $\qquad$ $\sigma : \mathbb{T}^{n-1} \longrightarrow \mathbb{C} \cong \mathbb{R}^2$

$\qquad\qquad\qquad\qquad (z_1, \dots, z_{n-1}) \longmapsto z_1 + \cdots + z_{n-1}$

If every pt of M_n is a regular pt of σ, then $\underline{\text{yes}}$.

$n=4$

$n=5$ — smooth