Embedded Submanifolds

Sunday, February 19, 2023 6:17 PM

Throughout M is a smooth manifold with or without bdy

Defn: $S \subset M$ is an embedded submanifold if

(1) S is a topological manifold (w/o bdy) with subspace topology.

(2) S has smooth structure s.t $i: S \hookrightarrow M$ is a smooth embedding

i.e. a smooth immersion & topological embedding.

Ex:

NonEx:

NOT embedded b/c not a manifold in subspace topology.

Terminology: $S \subset M$ embedded

(1) $\dim M - \dim S$ is the codimension of S in M.

(2) If $\dim M - \dim S = 1$, S is a hypersurface in M.

(3) M is called the ambient manifold/space.

Prop: $S \subset M$ embedded codim 0 $\iff$ S open

Idea: ($\Leftarrow$) S has subspace top & smooth structure from restricting charts on M.

$\therefore$ Coord rep $i: S \hookrightarrow M$ is Id.

$\therefore$ i is smooth immersion

($\Rightarrow$) $i: S \hookrightarrow M$ smooth emb

$\Rightarrow$ local diffeo $\leftarrow$ (b/c codim 0. $\Rightarrow$ immersion = local diffeo)

$\Rightarrow$ open map

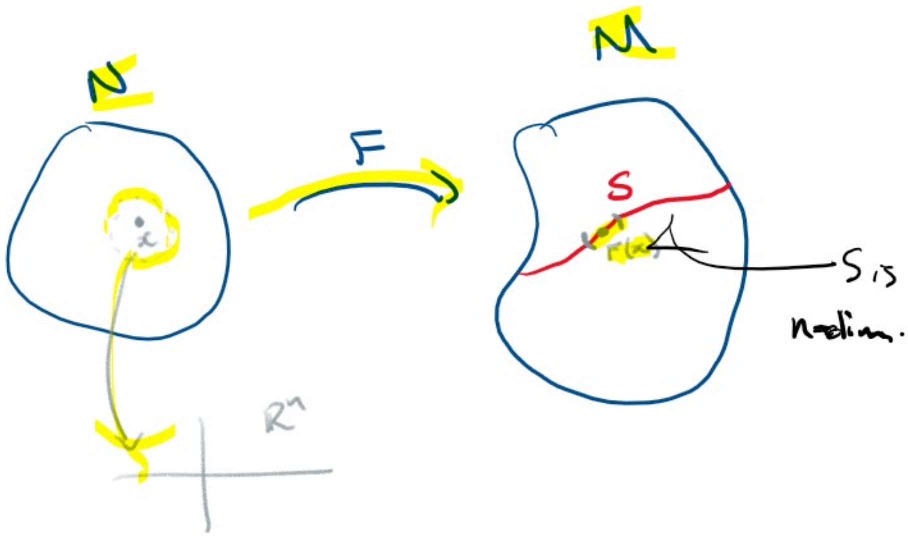
$\Rightarrow$ S open in M.

$\#$

What about codim $S > 0$?

Prop: N smooth, $F: N \to M$ smooth embedding
$S = F(N)$ w/ subspace top. Then
(1) S is topological manifold w/ unique smooth structure s.t. it is embedded in M
(2) F is a diffeo onto its image.

Idea: F embedding $\Rightarrow$ S top manifold.
Give S smooth structure w/ charts $(F(U), \varphi F^{-1})$ where (U, φ) is chart for N.
Check $S \hookrightarrow M$ is smooth embedding by looking at
$S \xrightarrow{F^{-1}} N \xrightarrow{F} M$. #



Example:
(Graphs as submanifolds)
N smooth manifold
$U \subset N$ open. $f: U \to M$ smooth.
$\Gamma(f) = \{(x,y) \in N \times M \mid f(x) = y, x \in U\} \subset N \times M$.
is embedded in $N \times M$.

Idea: $\gamma_f: U \to N \times M$, $\gamma_f(x) = (x, f(x))$ is smooth embedding.

$d\pi \circ d\gamma_f = d\,Id \Rightarrow \gamma_f$ immersion
$\Rightarrow$ π cts $\Rightarrow \gamma_f$ top embedding.
& π inverse for γ_f.

Slice Charts

Idea: $\mathbb{R}^n$ has chart $(\mathbb{R}^n, Id_{\mathbb{R}^n})$.

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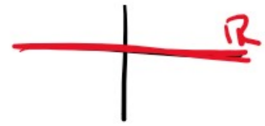

Coord functions $x^i(p^1,\dots,p^n) = p^i$

Embed $\mathbb{R}^k \subset \mathbb{R}^n$ as subset $\{(x^1,\dots,x^k,0,\dots,0)\}$.
and first k-coord functions are chart for $\mathbb{R}^k$.

Want similar charts for $S \subset M$ embedded
So we can work locally w/ embedded submanifolds while "remembering" how S is contained in M topologically

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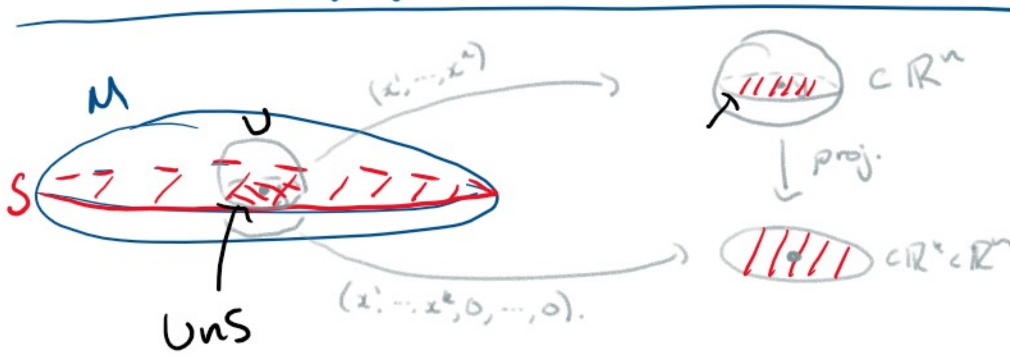


Defn: $V \subset \mathbb{R}^n$ open. A k-slice of V is a subset
$$S = \{(x^1,\dots,x^k,c^{k+1},\dots,c^n) \mid c^i = \text{const}\}.$$

Defn: $S \subset M$ subset satisfies the local k-slice condition if $\forall s \in S$ there is a smooth chart (U,φ) for M s.t. $\varphi(S \cap U)$ is a single k-slice of $\varphi(U)$.

Theorem: $S \subset M$ embedded k-dim. Then

Theorem: $S \subset M$ embedded k-dim. Then S satisfies local k-slice condition

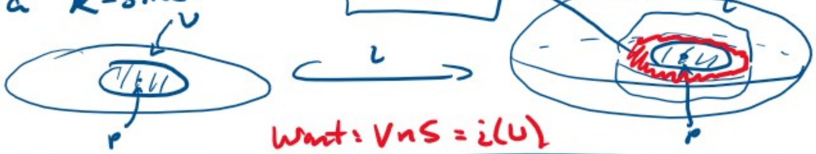
Proof: $i : S \hookrightarrow M$ immersion

Rank theorem $\Rightarrow$ find charts (U, φ) for S, (V, ψ) for M, $i(U) \subset V$.

s.t. $\hat{i} : U \to V$ is $(x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0)$

Problem: V might be "too big" so $i(U) \subset V$ may not be a k-slice.

$\underbrace{V \cap S \supsetneq i(U)}$

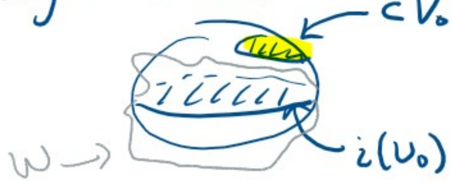


Want: $V \cap S = i(U)$

Soln: Shrink both sets to coord balls centered at p, so that i is inclusion

Still could have $V_0 \cap S \neq U_0$ i.e. $V_0 \cap S$ is not a single k-slice.

eg.



S has subspace top.

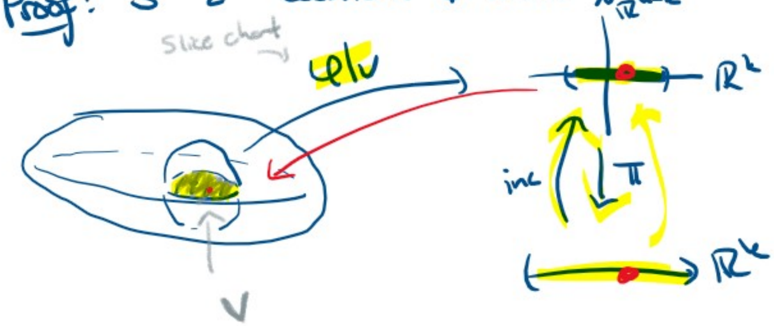
$\Rightarrow U_0 = W \cap S$ for some $W \subset M$ open.

Take $V_1 = W \cap V_0$ to make $V_1 \cap S$ a single k-slice. Then $(V_1, \psi|_{V_1})$ is a slice chart for S. $\square$

Theorem: $S \subset M$ has local k-slice condition

Theorem: $S \subset M$ has local k-slice condition
w/ subspace topology, S is
(1) a topological manifold of dim k.
(2) has smooth structure s.t. $S \subset M$ embedded.

"Proof": S 2nd countable & Hausdorff ✓



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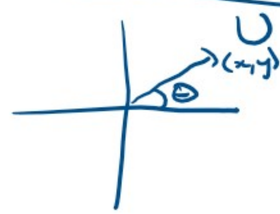
Ex: $U = \{(x,y) \mid x>0\} \subset \mathbb{R}^2$



$\varphi: U \longrightarrow \mathbb{R}^2$

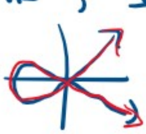
$(x,y) \longmapsto (\underbrace{x^2+y^2}_{\text{"}r\text{"}}, \underbrace{\arctan\left(\frac{y}{x}\right)}_{\text{"}\theta\text{"}})$

This is a slice chart for $S^1 \subset \mathbb{R}^2$.
Polar coords $\Rightarrow$ S^1 embedded submanifold of $\mathbb{R}^2$. Still needs multiple charts to define S^1
Let's do better!

Level Sets!

Defn: $\Phi: M \to N$, $c \in N$. $\Phi^{-1}(c)$ is a level set of Φ.

Ex: $\Phi: \mathbb{R}^2 \to \mathbb{R}$, $\Phi(x,y) = x^2+y^2$
$S^1 = \Phi^{-1}(1)$.

Ex: $\Phi: \mathbb{R}^2 \to \mathbb{R}$, $\Phi(x,y) = y^2 - x^2(x+1)$.
$\Phi^{-1}(0) =$  not a manifold!!

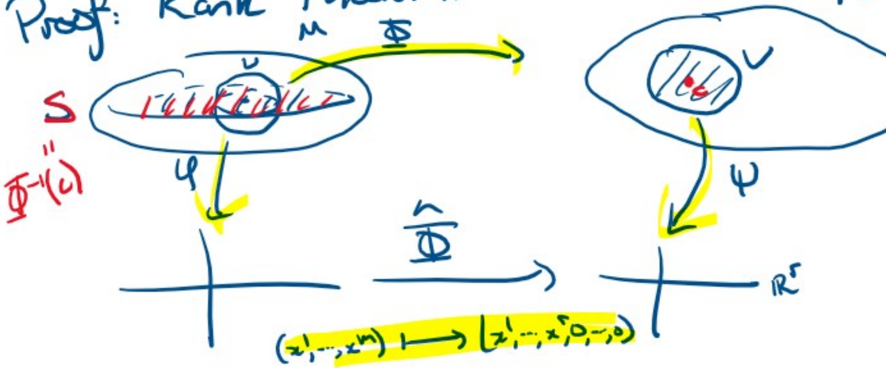
Q: Which level sets of $\Phi: M \to N$ are manifolds?

Theorem: M, N smooth manifolds

$\Phi: M \to N$ const rank r

Every level set of Φ is $\underbrace{\text{(properly)}}_{i: S \hookrightarrow M \text{ is proper.}}$ embedded of codim r.

Proof: Rank theorem.



So $S \cap U = \{(0, 0, \dots, x^{r+1}, \dots, x^m) \in U\}$.

$\Rightarrow$ S has local $(m-r)$-slice condition.

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Cor: Φ submersion $\Rightarrow$ every level set of Φ is (properly) embedded.

Proof: Every submersion has const rank = dim of codomain

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Defn: $\Phi: M \to N$

$p \in M$ is a regular pt if $d\Phi_p$ surjective

critical pt otherwise

... value if $\forall p \in \Phi^{-1}(c)$ is regular pt.

$f: \mathbb{R} \to \mathbb{R}$ smooth

$df_r : T_r\mathbb{R} \to T_{f(r)}\mathbb{R}$

1×1

$\|$ $\|$

critical pt   otherwise

$c \in N$ is a regular value if $\forall p \in \Phi^{-1}(c)$ is regular pt.
critical value otherwise

$\Phi^{-1}(c)$ is a regular level set if c is reg value.

1×1 $df_r : T_r\mathbb{R} \to T_{f(r)}\mathbb{R}$

$f' : \mathbb{R} \longrightarrow \mathbb{R}$

r is critical pt.

$f'(r) = 0$

Cor: Every regular level set is (properly) embedded.

Proof: $c \in N$ regular value. $\forall p \in \Phi^{-1}(c)$ to be regular pt.

$U = \{p \in M \mid \operatorname{rank} d\Phi_p = \dim N\}$ is open

(b/c $d\Phi_p$ surj $\Rightarrow$ nbhd V of p where $\Phi|_V$ submersion.)

so $\Phi|_U$ is a submersion & c regular value

$\Rightarrow \Phi^{-1}(c) \subset U$. $\Phi : U \to N$.

So $\Phi^{-1}(c)$ embedded in U

U open $\Rightarrow U$ codim 0 embedded in M

$\Rightarrow \Phi^{-1}(c)$ embedded in M.

$\square$

$\Phi^{-1}(c) \hookrightarrow U \hookrightarrow M$

$\mathbb{R} \longrightarrow \mathbb{R}^2$
$t \longmapsto (t^2, t^3)$.