17. II. 23

Smooth Covering Maps

$\pi : E \longrightarrow M$ is a smooth covering map when it is smooth, surjective, each pt of M has a nbhd U s.t. components of $\pi^{-1}U$ are mapped diffeomorphically onto U.

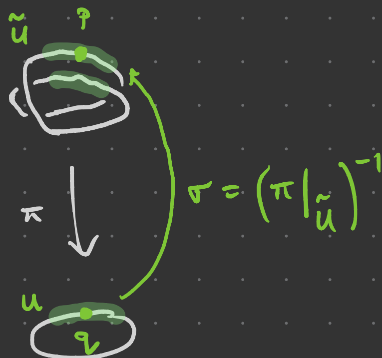
Prop (a) Smooth coverings are local diffeos, smooth submersions, open, & quotient maps.

(b) Injective smooth coverings are diffeos

(c) Top'l covering is a smooth covering iff it's a local diffeo

E.g. $\varepsilon : \mathbb{R} \longrightarrow S^1$, $\varepsilon^n : \mathbb{R}^n \longrightarrow \mathbb{T}^n$, $q : S^n \longrightarrow \mathbb{RP}^n$

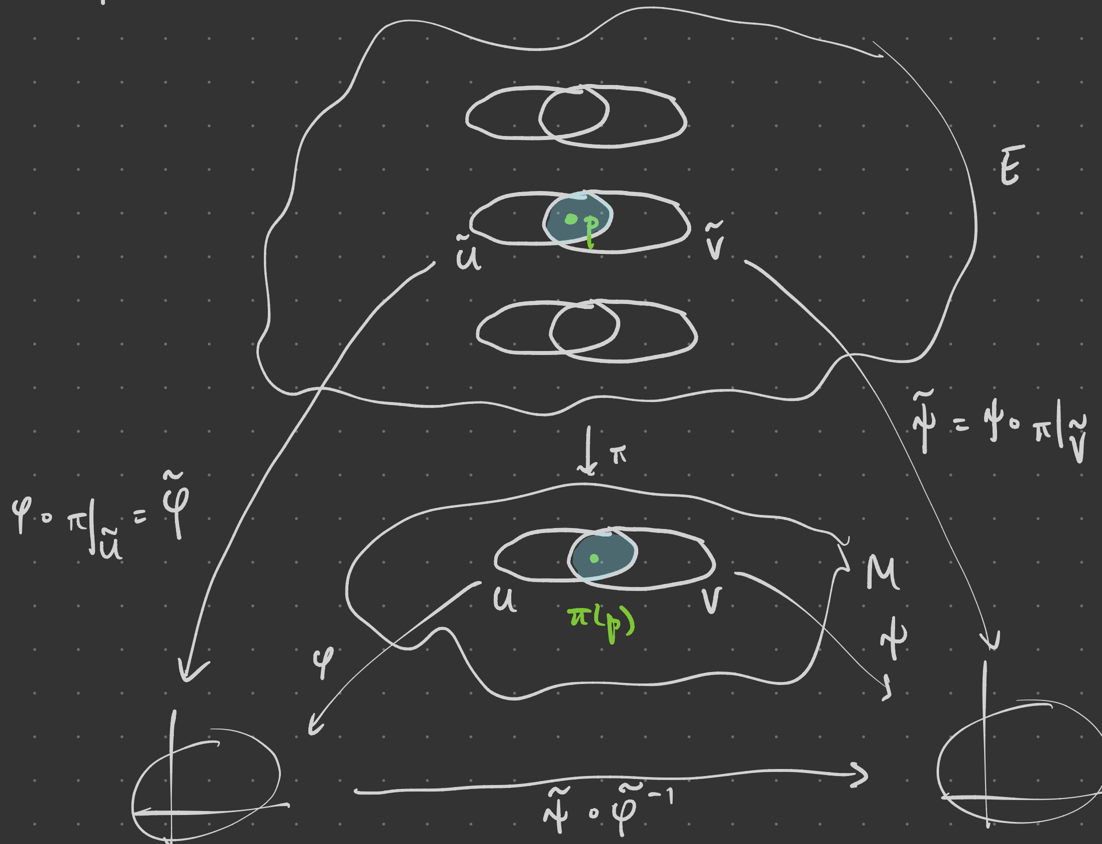
Note For $\pi : E \to M$ smooth covering, $U \subseteq M$ open & evenly covered, $q \in U$, $p \in \pi^{-1}\{q\}$ $\exists!$ smooth local section
$$\sigma : U \to E, \quad q \mapsto p$$



Prop M conn'd smooth n-mfld, $\pi : E \to M$ top'l covering map. Then E is a top'l n-mfld and has a unique smooth structure s.t. π is a smooth covering map.

Pf Top'l mfld p.93 (locally Euclidean b/c π local homeomorphism).

For smooth structure, given $p \in E$ take $U \subseteq M$ evenly covered nbhd of $\pi(p)$ and domain of smooth coord map $\varphi : U \longrightarrow \mathbb{R}^n$.



$$= (\psi \circ \pi|_{\tilde{U}\cap\tilde{V}}) \circ (\varphi \circ \pi|_{\tilde{U}\cap\tilde{V}})^{-1}$$

$$= \psi \circ (\pi|_{\tilde{U}\cap\tilde{V}} \circ \pi|_{\tilde{U}\cap\tilde{V}}^{-1}) \circ \varphi^{-1}$$

$$= \psi \circ \varphi^{-1} \quad \text{— smooth!}$$

Coordinate presentation of $\pi|_{\tilde{U}}$ in terms of $(\tilde{U}, \tilde{\varphi})$, (U, φ) is the identity map, so smooth covering map. $\square$

Cor Conn'd smooth mflds admit (unique) universal smooth covers.

Fact $\underbrace{\text{Proper}}$ local diffeos are smooth covering maps. p.95

$\pi : E \to M$, $\pi^{-1}K$ compact $\forall K \subseteq M$ compact. E.g. E compact!

Revisiting rank thm:

$\begin{cases} \text{constant rank} \rightsquigarrow \text{canonical form} \\ \end{cases}$

Rank Thm

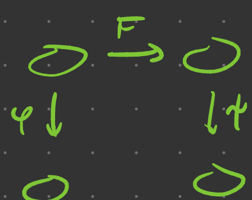
M, N smooth mflds of dimn m, n

$F : M \longrightarrow N$ smooth of constant rank r

Then $\forall p \in M$ $\exists$ smooth charts (U, φ) for M centered at p

(V, ψ) for N centered at $F(p)$

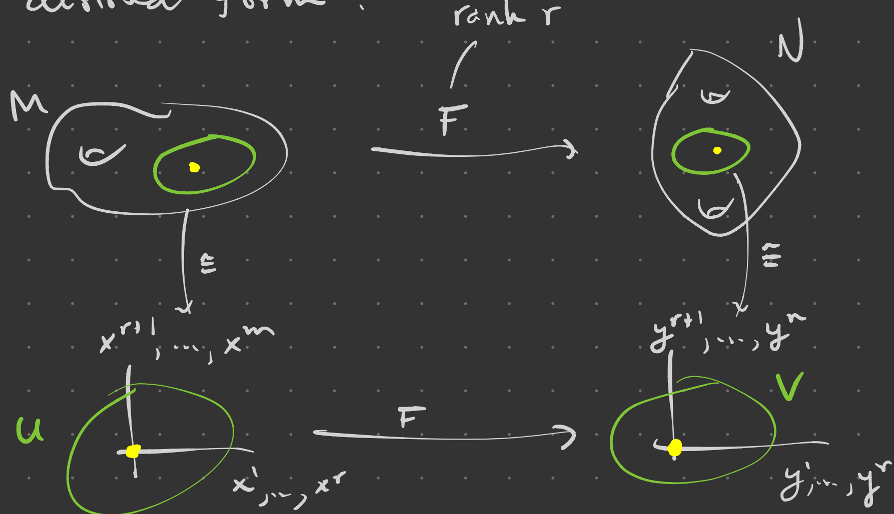
s.t. $FU \subseteq V$ in which F has coord rep'n



$$\hat{F}(x^1, \dots, x^r, x^{r+1}, \dots, x^m) = (x^1, \dots, x^r, 0, 0, \dots, 0).$$

$\psi F \varphi^{-1}$

Discussion Proof must produce (U, φ), (V, ψ) so that $\psi F \varphi^{-1}$ has desired form.



WLOG, U, V open in $\mathbb{R}^m$, $\mathbb{R}^n$, $p = 0$, $F(p) = 0$.

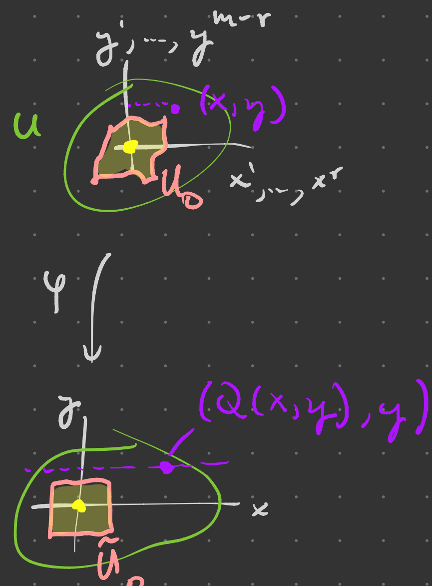
Further assume $JF(0)$ has upper left $r \times r$ submatrix nonsingular.

$F = (Q, R)$ for $Q : U \longrightarrow \mathbb{R}^r$, $R : U \longrightarrow \mathbb{R}^{n-r}$

Define $\varphi : U \longrightarrow \mathbb{R}^m$

$$(x,y) \longmapsto (Q(x,y), y)$$

so that $J\varphi(0) = \left(\begin{array}{c|c} \frac{\partial Q^i}{\partial x^j}\Big|_0 & \frac{\partial Q^i}{\partial y^j}\Big|_0 \\ \hline 0 & I \end{array}\right)$



By Inv Fn Thm, have $U_0 \subseteq U$ open $\ni 0$

s.t. $\varphi|_{U_0} : U_0 \xrightarrow{\cong} \tilde{U}_0$. Shrink $\tilde{U}_0$ to a cube

$$\varphi^{-1}(x,y) = (A(x,y), B(x,y))$$
$$= (A(x,y), y)$$

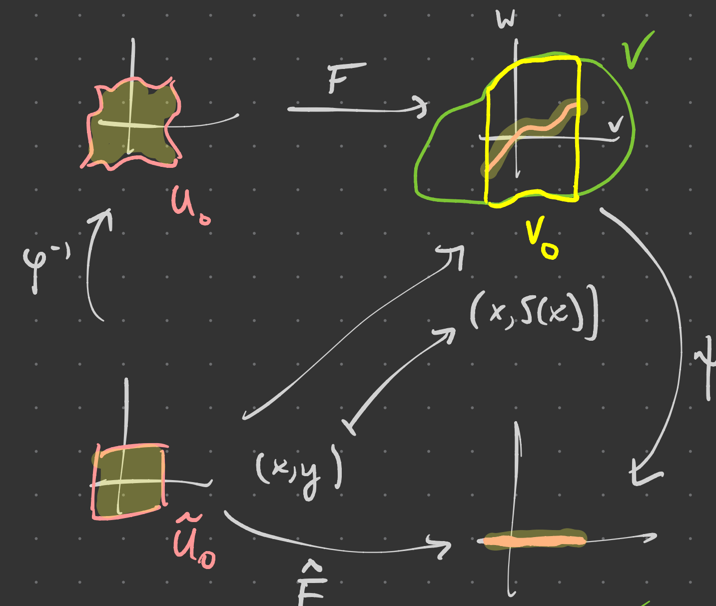
$$F \circ \varphi^{-1}(x,y) = (x, \underbrace{R(A(x,y), y)}_{\tilde{R}(x,y)})$$

$\tilde{R}(x,y) = \tilde{R}(x,0)$ $\forall y$ $=: S(x)$

For this,

$$J(F \circ \varphi^{-1}) = \left(\begin{array}{c|c} I & 0 \\ \hline \frac{\partial \tilde{R}}{\partial x} & \underbrace{\frac{\partial \tilde{R}}{\partial y}}_{0} \end{array}\right)$$

So $F \circ \varphi^{-1}(x, y) = (x, S(x))$



$(x, y) \longmapsto (x, 0)$ ✓

Define $V_0 = \{(v, w) \in V \mid (v, 0) \in \tilde{U}_0\}$

$\psi : V_0 \longrightarrow \mathbb{R}^n$

$(v, w) \longmapsto (v, w - S(v))$

□

Let's check this for rank 1 map $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$(x,y) \longmapsto (\sin(x+y), \cos(x+y)-1)$$

$(0,0) \longmapsto (0,0)$ ✓

$$JF = \begin{pmatrix} \cos(x+y) & \cos(x+y) \\ -\sin(x+y) & -\sin(x+y) \end{pmatrix}$$

rank 1 ✓

$$JF(0,0) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

as expected

$\varphi(x,y) = (\sin(x+y), y)$ and $\varphi^{-1}(x,y) = (\arcsin(x) - y, y)$ for $-1 < x < 1$

$$F \circ \varphi^{-1}(x,y) = (x, \cos(\arcsin(x)) - 1)$$
$$= (x, \underbrace{\sqrt{1-x^2} - 1}_{S(x)})$$

Set $\psi(v,w) = (v, w - \sqrt{1-v^2} + 1)$

Then $\psi \circ F \circ \varphi^{-1}(x, y) = \psi(x, \sqrt{1-x^2} - 1) = (x, 0)$ ✓