15. II. 23

Global Rank Thm $F: M \to N$ smooth of constant rank

(a) F surj $\Rightarrow$ F smooth sub

(b) F inj $\Rightarrow$ F smooth imm

(c) F bij $\Rightarrow$ F diffeo

Pf (a) Prove the contrapositive by Baire category thm!

(b) Assume rank $r < m = \dim M$. By rank thm, have a local rep'n $(0, \varepsilon) \mapsto (0, 0)$ for all small $\varepsilon > 0$ so F not inj.

(c) By a, b F is a local diffeo. Also bij so a diffeo. $\square$

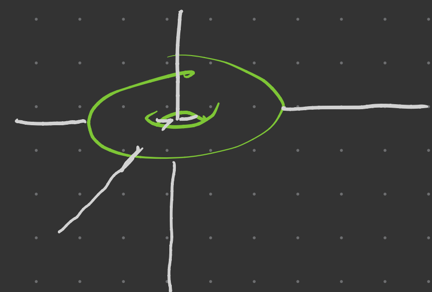
Smooth Embeddings

$F: M \longrightarrow N$ smooth is a smooth embedding when it is a smooth immersion and topological embedding (homeo onto its image).

E.g.

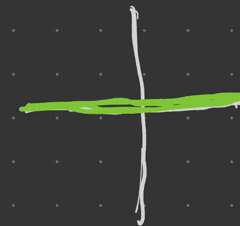
- $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k}$
  $x \longmapsto (x, 0)$

- $$\begin{tikzcd} \mathbb{R}^2 \arrow[r] \arrow[d] & \mathbb{R}^3 \\ \mathbb{T}^2 \arrow[ur, hook] & \end{tikzcd}$$

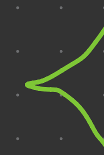


Non-e.g.

- $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$
  $t \longmapsto (t^3, 0)$

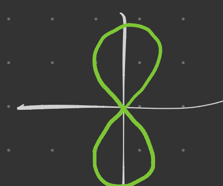


$t \mapsto (t^2, t^3)$



top'l emb and smooth map, but not an immersion since $\gamma'(0) = (0,0)$.

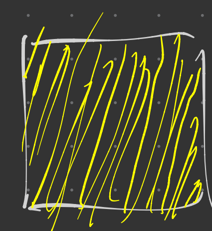
- $\beta : \mathbb{R} \longrightarrow \mathbb{R}^2$
  $t \longmapsto (\sin 2t, \sin t)$



smooth imm but not embedding

- $\gamma : \mathbb{R} \longrightarrow \mathbb{T}^2$
  $t \longmapsto (\exp(2\pi i t), \exp(2\pi i \alpha t))$

  α irrational



Have γ smooth immersion and injective, but $\gamma(\mathbb{Z})$ has $\gamma(0)$ as a limit point* while $\mathbb{Z} \subseteq \mathbb{R}$ has no limit points $\Longrightarrow$ γ not an embedding.

* Consequence of Dirichlet's approximation theorem.

Q When is an injective smooth immersion a smooth embedding?

Prop M, N smooth mflds w/ or w/o ∂, $F: M \longrightarrow N$ inj smooth imm. If any of the following holds, then F is a smooth embedding:

(a) F open or closed
(b) F proper
(c) M compact
(d) $\partial M = \varnothing$ and $\dim M = \dim N$

Pf (a) F open or closed $\Rightarrow$ F top'l emb ✓

(b), (c) $\Rightarrow$ F closed ✓

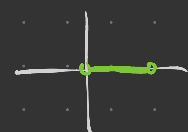
(d) dF_p nonsingular $\forall p$, $F(M) \subseteq N^\circ$

$F : M \longrightarrow N^\circ$ is a local diffeo $\Rightarrow$ open

and $M \longrightarrow N^\circ \hookrightarrow N$ is open, so F top'l emb. ✓ $\square$

Note $\exists$ smooth embeddings which are neither open nor closed.

E.g. $$\begin{aligned} (0,1) &\hookrightarrow \mathbb{R}^2 \\ x &\longmapsto (x,0) \end{aligned}$$



has image neither open nor closed.

Local embedding thm M, N smooth mflds w/ or w/o ∂, $F: M \to N$ smooth.

Then F is a smooth immersion iff every pt in M has a nbhd $U \subseteq M$

s.t. $F|_U : U \to N$ is a smooth embedding.

Pf $\Leftarrow$: Embeddings have full rank.

$\Rightarrow$: F sm imm, $p \in M^\circ$. By rank thm $\exists$ nbhd U_1 of p on which

$\hat{F} : (x^1, \dots, x^m) \longmapsto (x^1, \dots, x^m, 0, \dots, 0)$. Thus $F|_{U_1}$ inj.

Take precompact nbhd U of p with $\bar{U} \subseteq U_1$.

$F|_{\bar{U}}$ inj cts w/ compact domain, so closed map lemma

$\Rightarrow F|_{\bar{U}}$ top'l emb $\Rightarrow F|_U$ top'l emb.

$p \in \partial M$: p. 87 to produce U_1. $\square$

Defn A cts map $F: X \to Y$ is a topological immersion if it is locally an embedding.

Submersions

$$\begin{tikzcd} M \arrow[d, "\pi"'] \\ N \arrow[u, dashed, bend right, "\sigma"'] \end{tikzcd}$$

section of π when $\pi \circ \sigma = id_N$.

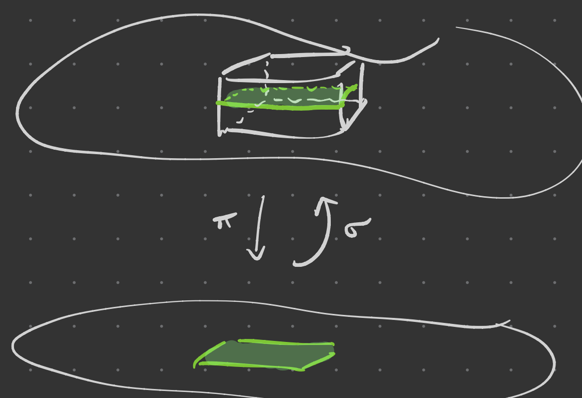
A local section of π is a cts map $\sigma : U \to M$ defined on some $U \subseteq N$ open s.t. $\pi \circ \sigma = id_U$.

Local Section Thm $\pi : M \to N$ smooth. Then π is a smooth submersion iff every pt of M is in the image of a smooth local section of π.

Pf $\Leftarrow$: Suppose $p \in M$ and $\sigma : U \to M$ is a smooth local section with $\sigma(q) = p$, $q = \pi(\sigma(q)) = \pi(p) \in N$. Since $\pi \circ \sigma = id_U$,

$d\pi_p \circ d\sigma_q = id_{T_qN} \implies d\pi_p$ surjective.

$\Rightarrow$: Rank thm + $(x^1, \dots, x^m) \longmapsto (x^1, \dots, x^r)$ admits section $(x^1, \dots, x^r) \longmapsto (x^1, \dots, x^r, 0, \dots, 0)$



$\square$

Dfn $\pi : X \longrightarrow Y$ cts map is a topological submersion if every point of X is in the image of a cts local section of π.

Prop Smooth submersions are open; smooth surj submersions are quotient maps.

Pf

$$\begin{array}{ccc} W \subseteq M & \xrightarrow{\pi} & N \\ \in & & \ni \\ p & \xleftarrow{\text{open}} \quad \mapsto & q \\ & \text{sm sub} & \end{array}$$

Take U open nbhd of q on which $\exists$ local section $\sigma : U \longrightarrow M$ s.t. $\sigma(q) = p$.

$$\begin{tikzcd} p \arrow[d, mapsto] & M \arrow[d, "\pi"'] & p \; \supseteq W \\ q & N \supseteq U \arrow[u, bend right, "\sigma"'] & q \; \supseteq \sigma^{-1}W \arrow[u, mapsto] \end{tikzcd}$$

If $y \in \sigma^{-1}W$, then $y = \pi(\sigma(y)) \in \pi W$. Thus $\sigma^{-1}W$ is a nbhd of q contained in πW

$\Rightarrow \pi W$ open $\Rightarrow \pi$ open.

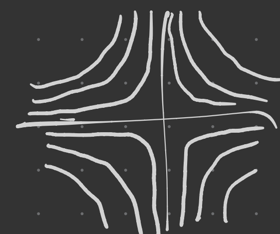
Surj + open $\Rightarrow$ quotient. $\square$

Surjective smooth submersions are a large and important class of quotients in Diff.

$\exists$ quotients in Diff that are <u>not</u> surj sm subs.

E.g. $\mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x,y) \longmapsto xy$



<u>Thm</u> $\pi : M \longrightarrow N$ surj sm sub. Then $\forall$ sm mfld P w/ or w/o ∂, $F : N \longrightarrow P$ is smooth iff $F \circ \pi$ is smooth:

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "F \circ \pi"] & \\ N \arrow[r, "F"'] & P \end{tikzcd}$$

$\square$

Thm $\pi: M \longrightarrow N$ surj sm sub, P mfld w/ or w/o ∂,

$F: M \longrightarrow P$ smooth & constant on fibers of π, then $\exists!$ sm map

$\tilde{F}: N \longrightarrow P$ s.t. $\tilde{F} \circ \pi = F$:

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "F"] & \\ N \arrow[r, dashed, "\tilde{F}"'] & P \end{tikzcd}$$

$\square$

Thm

$$\begin{tikzcd} & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ N_1 \arrow[rr, dashed, "\exists! \approx"'] & & N_2 \end{tikzcd}$$

π_1, π_2 surj sm subs with same fibers then $\exists!$ diffeo

$F: N_1 \longrightarrow N_2$ s.t. $F \circ \pi_1 = \pi_2$.

$\square$

$$
\begin{tikzcd}
\mathbb{R}^{n+1} \setminus 0 \arrow[r, "F"] \arrow[d, "\pi"'] \arrow[dr] & \mathbb{R}^{k+1} \setminus 0 \arrow[d] \\
\mathbb{RP}^n \arrow[r] & \mathbb{RP}^k
\end{tikzcd}
$$

$$[x] \longmapsto [F(x)]$$

$$F(\lambda x) = \lambda^d F(x)$$

F smooth

smooth as long as $\mathbb{R}^{n+1} \setminus 0 \longrightarrow \mathbb{RP}^n$

sm surj sub