13. II. 23

Maps of constant rank

Idea

Smooth Maps $F: M \longrightarrow N$ $\xrightarrow{\text{linear approximation}}$ Differentials on Tangent Spaces $dF_p : T_pM \longrightarrow T_{F(p)}N$ $\xrightarrow{\text{rank}}$ Geometric Properties $\text{rank}(dF_p)$

For good local models, need constant rank dF_p, $p \in M$

- dF_p surjective $\forall p$: submersion
- dF_p injective $\forall p$: immersion

- immersion + homeo onto image : embedding
  - F is a

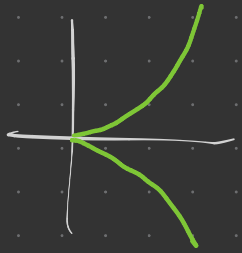
Given linear transformation $L : V \longrightarrow W$, rank(L) $:=$ dim im(L)

For smooth $F : M \longrightarrow N$ and $p \in M$, $\text{rank}_p(F) := \text{rank}\, dF_p$

If $p \longmapsto \text{rank}_p F$ is constant, say F has constant rank and $\text{rank}\, F := \text{rank}_p F$ (for some/any $p \in M$).

E.g.

$$F : \mathbb{R} \longrightarrow \mathbb{R}^2$$
$$t \longmapsto (t^2, t^3)$$



$$dF_p : \mathbb{R} \longrightarrow \mathbb{R}^2$$
$$t \longmapsto (2pt, 3p^2 t)$$

$$\text{rank}_p F = \begin{cases} 1 & \text{if } p \neq 0 \\ 0 & \text{if } p = 0 \end{cases}$$ — not of constant rank.

Note $\operatorname{rank}_p F \leq \min\{\dim M, \dim N\}$

say F has full rank at p when equal

Smooth submersion : $\operatorname{rank} F = \dim N$

Smooth immersion : $\operatorname{rank} F = \dim M$

Prop Suppose $F : M \longrightarrow N$ smooth, $p \in M$. If dF_p is surjective (injective) then $\exists$ nbhd U of p s.t. $F|_U$ is a submersion (immersion)

Pf Sketch $(\text{full rank}) \subseteq \mathbb{R}^{m \times n}$ is open

Q Examples of immersions, submersions, embeddings?

(1) $$\begin{tikzcd} \varepsilon : \mathbb{R} \arrow[r] & S^1 \\ (0,1) \arrow[u, phantom, "\cup|"] \arrow[ur] & \end{tikzcd}$$ imm + sub


embedding

(2) $S^1 \times S^1 \xrightarrow{\quad} S^1$ submersion

p_1

$(z, w) \longmapsto z$

(3) $\mathbb{R} \hookrightarrow \mathbb{R}^2$ embedding

$S^1 \times S^1 \hookrightarrow \mathbb{R}^3$

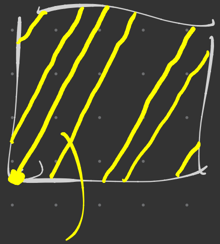
$\cap|$

$\mathbb{C} \times \mathbb{C}$

(4) $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1}$

$[x] \longmapsto [x, 0]$

(5)


Imm ✓

irrational slope

Q Embedding?

TPS $(\text{immersion}) \circ (\text{immersion}) \overset{?}{=} \text{immersion}$, ✓

What about submersions? ✓ Maps of constant rank? ✗

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "\pi_2"] & \mathbb{R}^2 \arrow[r, "\pi_2"] & \mathbb{R}^2 \\
(x,y) \arrow[r, mapsto] & (0,y) & \\
 & (x,y) \arrow[r, mapsto] & (x,0]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r] \arrow[dr, "\cos"'] & S^1 \subseteq \mathbb{R}^2 \arrow[d, "\pi_1"] \\
 & \mathbb{R}
\end{tikzcd}$$

not const rk

Inverse Function Thm M, N smooth mflds, $F: M \longrightarrow N$ smooth.

If $p \in M$ has dF_p invertible, then $\exists$ conn'd nbhds U_0 of p, V_0 of $F(p)$ s.t. $F|_{U_0} : U_0 \longrightarrow V_0$ is a diffeo.

□

$F : M \longrightarrow N$ smooth is a **local diffeomorphism** when $\forall p \in M$ $\exists$ nbhd U of p s.t. $FU \subseteq N$ open and $F|_U : U \to FU$ is a diffeomorphism.

Prop (a) F is a local diffeo iff smooth imm & smooth sub.

(b) If $\dim M = \dim N$ & F is a smooth imm or sub then F is a local diffeo.

Pf (a) $\Longrightarrow$: Given $p \in M$, have nbhd U w/ $F|_U : U \xrightarrow{\cong} FU$

$\Longrightarrow$ dF_p a linear iso $\Longrightarrow$ rank $F = \dim M = \dim N$

$\Longrightarrow$ F smooth imm & sub.

$\Longleftarrow$: dF_p iso $\forall p$. Apply inverse fn thm.

(b) Since $\dim M = \dim N$, inj or surj of $dF_p \Rightarrow dF_p$ iso.

Thus F an imm $\iff$ a sub. $\square$

Rank Thm ⋰ $\left\{ \text{constant rank} \rightsquigarrow \text{canonical form} \right.$

M, N smooth mflds of dimn m, n

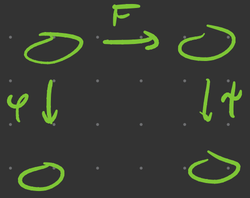
$F : M \longrightarrow N$ smooth of constant rank r

Then $\forall p \in M$ $\exists$ smooth charts (U, φ) for M centered at p

(V, ψ) for N centered at $F(p)$

s.t. $FU \subseteq V$ in which F has coord rep'n

$$\hat{F}(x^1, \dots, x^r, x^{r+1}, \dots, x^m) = (x^1, \dots, x^r, 0, 0, \dots, 0).$$



$\psi F \varphi^{-1}$

Note For F a smooth sub this becomes

$$\hat{F}(x^1, \dots, x^m) = (x^1, \dots, x^n) \qquad (m \geq n)$$

For F a smooth imm,

$$\hat{F}(x^1, \dots, x^m) = (x^1, \dots, x^m, 0, \dots, 0).$$

Pf WLOG

$$\begin{tikzcd} U \arrow[r, "F"] \arrow[d, phantom, "\cap"] & V \arrow[d, phantom, "\cap"] \\ \mathbb{R}^m & \mathbb{R}^n \end{tikzcd}$$

. Since rank $F = r$, $JF(p)$ has $r \times r$ submatrix w/ $\det \neq 0$. Reordering coords, may take this to be upper left corner of $JF(p)$.

Make coords $(x, y) = (x^1, \dots, x^r, y^1, \dots, y^{m-r})$ for U

$(v, w) = (v^1, \dots, v^r, w^1, \dots, w^{n-r})$ for V.

Furthermore, translate so $p = (0,0)$, $F(p) = (0,0)$.

Then $F(x,y) = (Q(x,y), R(x,y))$ for $Q : U \to \mathbb{R}^r$
$R : U \to \mathbb{R}^{n-r}$

and $\left(\underset{\substack{\shortparallel \\ C}}{\frac{\partial Q^i}{\partial x^j}\Big|_{(0,0)}} \right)_{i,j=1}^{r}$ is nonsingular.

Define $\varphi : U \to \mathbb{R}^m$
$(x,y) \mapsto (Q(x,y), y)$ with $J\varphi(0,0) = \left(\begin{array}{c|c} C & \overset{\left(\frac{\partial Q^i}{\partial y^j}\Big|_{(0,0)}\right)}{D} \\ \hline 0 & I_{n-r} \end{array} \right)$

nonsingular b/c det = det C.

By inverse fn thm, $\exists$ conn'd nbhds U_0 of $(0,0)$, $\tilde{U}_0$ of $\varphi(0,0)$ s.t. $\varphi : U_0 \to \tilde{U}_0$ diffeo.

Restrict so that $\tilde{U}_0$ is a cube. Write

$$\varphi^{-1}(x,y) = (A(x,y), B(x,y)) \text{ for } A:\tilde{U}_0 \to \mathbb{R}^r, \ B:\tilde{U}_0 \to \mathbb{R}^{m-r}$$

smooth. Then

$$\begin{aligned}(x,y) &= \varphi(A(x,y), B(x,y)) \\ &= (Q(A(x,y), B(x,y)), B(x,y))\end{aligned}$$

Thus $B(x,y) = y$ so $\varphi^{-1}(x,y) = (A(x,y), y)$.

Since $\varphi \circ \varphi^{-1} = id$, get $Q(A(x,y), y) = x$

$$\Rightarrow F \circ \varphi^{-1}(x,y) = (x, \tilde{R}(x,y))$$

for $\tilde{R} : \tilde{U}_0 \to \mathbb{R}^{n-r}$ given by $\tilde{R}(x,y) = R(A(x,y), y)$.

$$\text{Thus}\quad \underbrace{J(F\circ\varphi^{-1})\underset{\substack{\in\\ \tilde{U}_0}}{(x,y)}}_{\text{has rank } r\,:} = \begin{pmatrix} I_{n-r} & 0 \\ \underbrace{\dfrac{\partial \tilde{R}^i}{\partial x^j}\Big|_{(x,y)}}_{\substack{r \text{ lin ind}\\ \text{cols}}} & \underbrace{\dfrac{\partial \tilde{R}^i}{\partial y^j}\Big|_{(x,y)}}_{\text{all } 0} \end{pmatrix}$$

r lin ind cols $\Rightarrow$ all 0

$\Downarrow$

$\tilde{R}$ independent of $y^1, \dots, y^{m-r}$

$\text{rank}\,(F\circ \text{diffeo}) = \text{rank}\, F$

Let $S(x) = \tilde{R}(x,0)$. Then

$$F\circ\varphi^{-1}(x,y) = (x, S(x))$$

Finally, need smooth chart in nbhd $(0,0)\in V$.

Define $V_0 = \{(v,w) \in V \mid (v,0) \in \tilde{U}_0\}$. Then $F \circ \varphi^{-1}(\tilde{U}_0) \subseteq V_0$

$\Rightarrow F(U_0) \subseteq V_0$. Define $\psi : V_0 \xrightarrow{\cong} \mathbb{R}^n$

$$(v,w) \longmapsto (v, w - S(v))$$

diffeo w/ inverse $\psi^{-1}(s,t) = (s, t + S(s))$.

Now get $\psi \circ F \circ \varphi^{-1}(x,y) = \psi(x, S(x)) = (x, S(x) - S(x)) = (x, 0)$.

$\square$

Cor $F : M \longrightarrow N$ smooth, M conn'd. Then TFAE:

(a) $\forall p \in M$ $\exists$ smooth charts containing p and $F(p)$ in which the coord rep'n of F is linear

(b) F has constant rank $\square$