10.II.23

Curves and Velocity

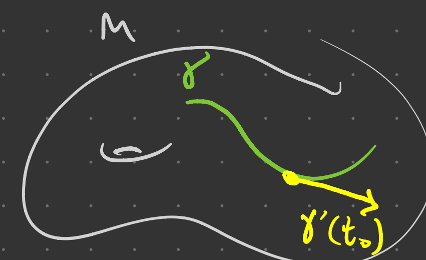
$J \subseteq \mathbb{R}$ an interval (open or closed)

M smooth mfld

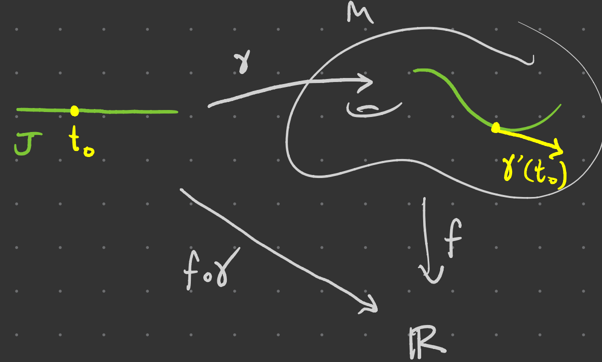
A curve in M is a cts map $\gamma : J \to M$

For a smooth curve $\gamma : J \to M$ and $t_0 \in J$, then velocity of γ at t_0 is

$$\gamma'(t_0) = d\gamma\left(\left.\frac{d}{dt}\right|_{t_0}\right) \in T_{\gamma(t_0)} M$$



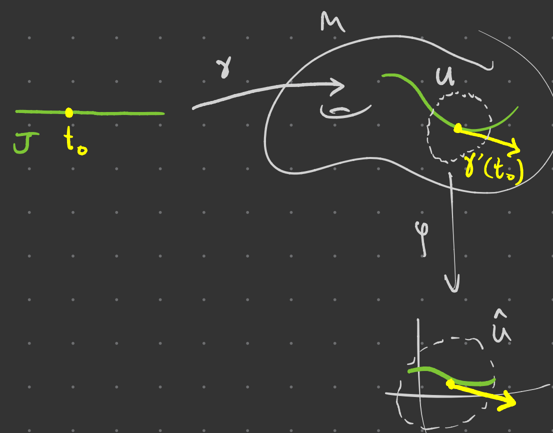
For $f \in C^\infty(M)$, $\gamma'(t_0)f = d\gamma\left(\left.\frac{d}{dt}\right|_{t_0}\right)f = \left.\frac{d}{dt}\right|_{t_0}(f \circ \gamma) = (f \circ \gamma)'(t_0)$



If t_0 is an endpt of J, just use 1-sided limit to eval $(f\circ\gamma)'(t_0)$.

For (U,φ) a smooth chart w/ coord fns (x^i):

if $\gamma(t_0)\in U$, then $\gamma(t) = \left(\gamma^1(t), \dots, \gamma^n(t)\right)$ for t close to t_0



$\gamma^i = x^i\circ\gamma$

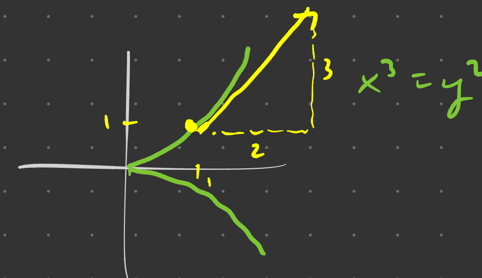
so this really means

$\gamma(t) = \varphi^{-1}(\varphi\circ\gamma)$:)

Thus $\gamma'(t_0) = \frac{d\gamma^i}{dt}(t_0) \frac{\partial}{\partial x_i}\Big|_{\gamma(t_0)}$.

E.g. $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$

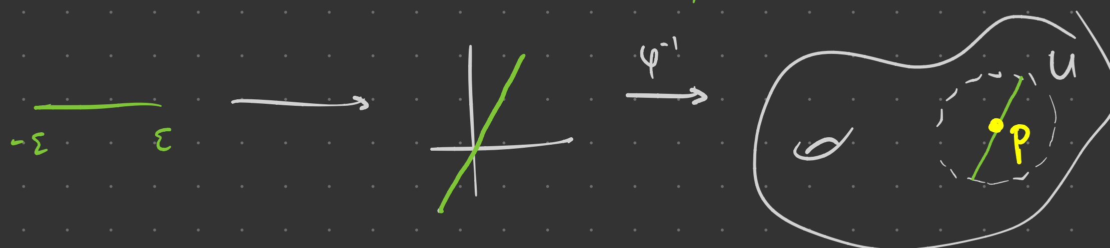
$t \longmapsto (t^2, t^3)$



$$\gamma'(t_0) = 2t_0 \frac{\partial}{\partial x}\Big|_{\gamma(t_0)} + 3t_0^2 \frac{\partial}{\partial y}\Big|_{\gamma(t_0)}$$

Prop Suppose M is a smooth mfld w/ or w/o ∂ and $p \in M$. Every $v \in T_pM$ is the velocity vector of some smooth curve in M.

Pf Suppose $p \in M^\circ$, let (U, φ) be a smooth coord chart centered at p, write $v = v^i \frac{\partial}{\partial x^i}\Big|_p$ in terms of coordinate basis. For small $\varepsilon > 0$, let $\gamma : (-\varepsilon, \varepsilon) \longrightarrow U$ be $\gamma(t) \underset{\varphi^{-1}}{=} (tv^1, \dots, tv^n)$



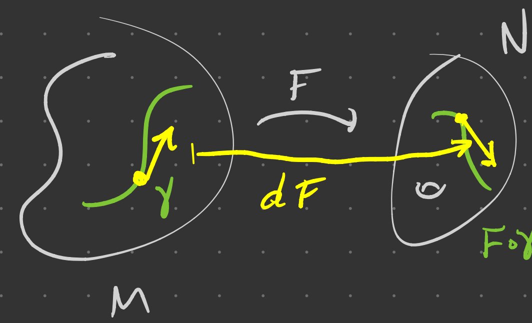
Then $\gamma(0) = p$, $\gamma'(0) = v^i \frac{\partial}{\partial x^i}\Big|_{\gamma(0)} = v$.

$p \in \partial M$: p. 70

$\square$

Prop (Velocity of a composite curve) $F: M \to N$ smooth, $\gamma: J \to M$ smooth curve. $\forall t_0 \in J$,

$$(F \circ \gamma)'(t_0) = dF(\gamma'(t_0)).$$



Pf Chain rule. $\square$

Cor (Computing diff'ls via velocities) $F: M \to N$ smooth, $p \in M$, $v \in T_pM$ then $dF_p(v) = (F \circ \gamma)'(0)$ for any smooth curve $\gamma: J \to M$ s.t. $0 \in J$, $\gamma(0) = p$, $\gamma'(0) = v$.

E.g. Take $g \in GL_n\mathbb{R} \subseteq \mathbb{R}^{n\times n}$. Left mult'n by g induces a smooth map $l_g : GL_n\mathbb{R} \longrightarrow GL_n\mathbb{R}$.

$$h \longmapsto gh$$

We may identify $T_e GL_n\mathbb{R} = T_g GL_n\mathbb{R} = \mathbb{R}^{n\times n}$.

Q What is $dl_g : T_e GL_n\mathbb{R} \longrightarrow T_g GL_n\mathbb{R}$ as a linear transformation $\mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$?

A For $X \in \mathbb{R}^{n\times n}$, choose $\gamma : J \longrightarrow GL_n\mathbb{R}$ with $\gamma(0) = e$, $\gamma'(0) = X$. Then $l_g(\gamma(t)) = g\gamma(t)$ and

$$d(l_g)_e = \frac{d}{dt}\Big|_{t=0} g\gamma(t) = g\gamma'(0) = gX.$$

Thus $d(l_g)_e : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$

$X \longmapsto gX$

linearity of $\frac{d}{dt}\Big|_{t=0}$ + local coords differentiation

is also left mult'n by g (but now on all $n\times n$ matrices).

Group work

(1) For $p=(x,y) \in \mathbb{R}^2$, define $\gamma_p : \mathbb{R} \longrightarrow \mathbb{R}^2$

$$t \longmapsto \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Compute $\gamma_p'(0)$.

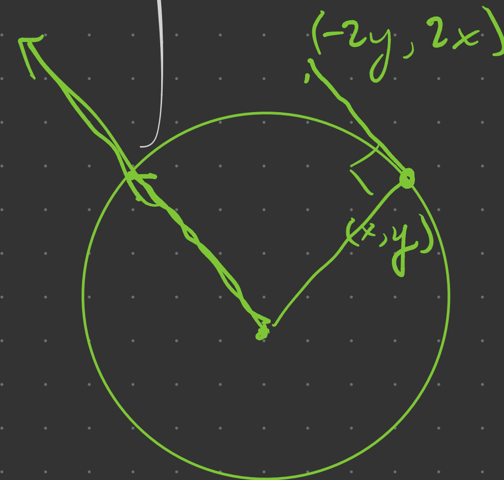
(2) Prove that a local maximum of a smooth fn $f : M \longrightarrow \mathbb{R}$ is a critical pt of f (i.e. $df_p = 0$ if $f(p)$ is a local max of f).

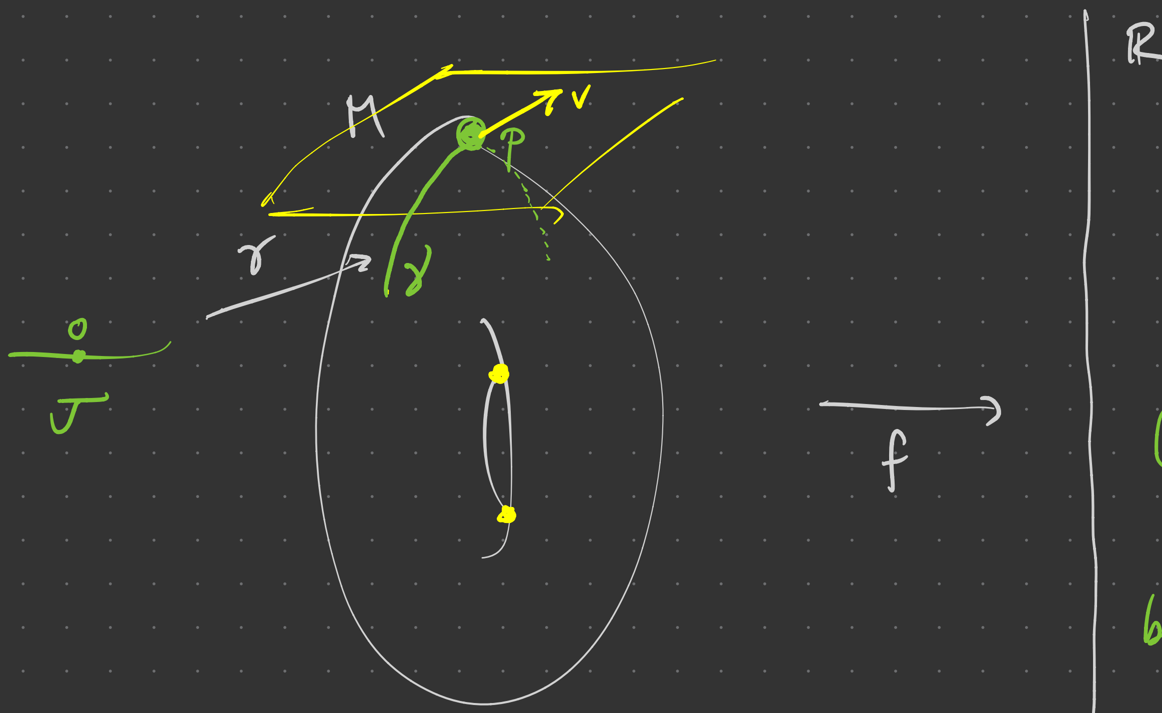
Hint Reduce to 1-D case using curves.

(1) Use component functions

$$t \longmapsto \begin{pmatrix} x\cos 2t - y\sin 2t \\ x\sin 2t + y\cos 2t \end{pmatrix} \begin{matrix} \longleftarrow \gamma_p^1 \\ \longleftarrow \gamma_p^2 \end{matrix}$$

$$\left.\frac{d}{dt}\right|_{t=0} : \begin{pmatrix} -2y \\ 2x \end{pmatrix}$$





Have

$$df_p(v) = (f\circ\gamma)'(0)$$

$(f\circ\gamma)(0)$ is a local max of $f\circ\gamma$ so by calc, $(f\circ\gamma)'(0) = 0$

WTS: $df_p(v) = 0 \quad \forall v$