Note True for $p \in \partial M$ too. pp. 57–58

8. II. 23

Prop V, W fin dim $\mathbb{R}$-vs, $L : V \to W$ linear then $\forall a \in V$

$$\begin{array}{ccc} v & \longmapsto & D_v|_a : f \longmapsto \frac{d}{dt}\Big|_{t=0} f(a+tv) \end{array}$$

$$\begin{tikzcd} V \arrow[r, "\cong"'] \arrow[d, "L"'] & T_a V \arrow[d, "dL_a"] \\ W \arrow[r, "\cong"] & T_{La} W \end{tikzcd}$$

$$w \longmapsto D_w|_{La} \quad \square$$

Note · $V \cong T_a V$ is "canonical".

· Natural transformation

$\mathrm{id}_{\mathrm{Vect}_*} \Rightarrow T_{\cdot}(\ )$

Thus for $M \subseteq V$ open submfld of an $\mathbb{R}$-vs, identify

$$T_p M, \ T_p V, \ \& \ V.$$

E.g. $T_A GL_n(\mathbb{R}) \cong \mathbb{R}^{n \times n}$ b/c $GL_n(\mathbb{R}) \subseteq \mathbb{R}^{n\times n}$ is an open submfld.

Computations in coordinates

For (U, φ) a smooth chart on $M \ni p$ have

$$d\varphi_p : T_pM \xrightarrow{\cong} \underbrace{T_{\varphi(p)}\mathbb{R}^n}_{\text{basis } \left.\frac{\partial}{\partial x^1}\right|_{\varphi(p)}, \dots, \left.\frac{\partial}{\partial x^n}\right|_{\varphi(p)}} \quad x^1, \dots, x^n$$

$$\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} = D_{e^i}|_{\varphi(p)}$$

Define $\left.\frac{\partial}{\partial x^i}\right|_p := d\varphi_p^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right)$ — coordinate vectors at p

(form a basis of T_pM)

Then for $f \in C^\infty(M)$, $\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} (f \circ \varphi^{-1}) = \frac{\partial \hat{f}}{\partial x^i}(\underbrace{\hat{p}}_{\varphi(p)})$

Every $v \in T_pM$ has a unique expression as

$$v = v^i \frac{\partial}{\partial x^i}\Big|_p . \quad \text{(Einstein summation)}$$

and $(v^1, \dots, v^n)$ are the components of v wrt the φ-coordinate basis.

We have $v(x^j) = \left(v^i \frac{\partial}{\partial x^i}\Big|_p \right)(x^j) = v^i \frac{\partial x^j}{\partial x^i}(p) = v^j .$

x^j — j-th component of φ

dF_p in coordinates pp. 61-63

For $F : U \longrightarrow V$, $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, with coordinates $x^1, \dots, x^n$ and $y^1, \dots, y^m$,

$$dF_p\left(\frac{\partial}{\partial x^i}\Big|_p \right) = \frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j}\Big|_{F(p)}$$

So in the coord bases, dF_p has matrix the Jacobian!

$$\begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix}$$

For $F : M \longrightarrow N$, dF_p is still represented by the Jacobian (wrt coordinate bases for charts at p, $\varphi(p)$ on M, N)!

Tangent Bundle

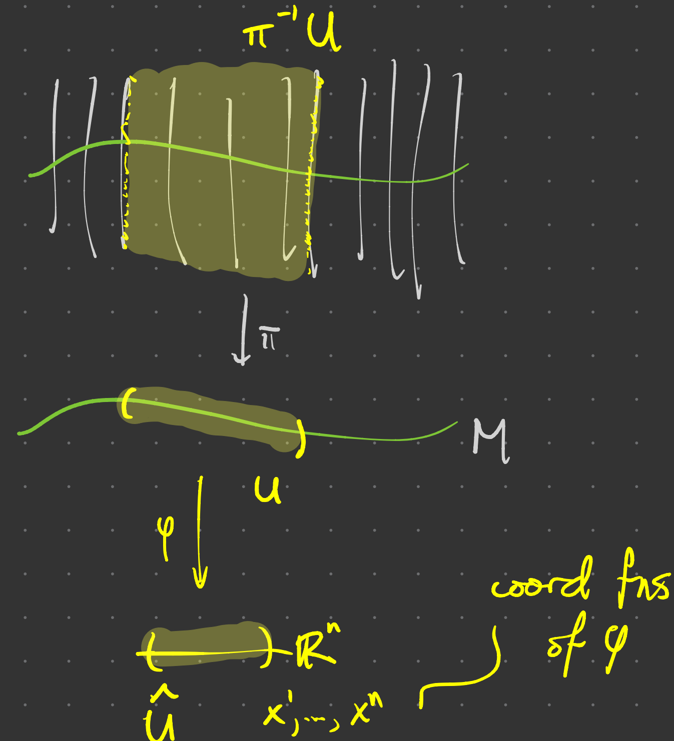
M smooth mfld w/ or w/o ∂

$$TM := \coprod_{p \in M} T_pM \xrightarrow[\pi]{} M$$

$$(p, v) \longmapsto p$$

For (U, φ) smooth chart on M,

define $\tilde{\varphi} : \pi^{-1}U \longrightarrow \mathbb{R}^{2n}$



$$\left(p, v^i \frac{\partial}{\partial x^i}\Big|_p\right) \longmapsto \left(x^1(p), \dots, x^n(p), v^1, \dots, v^n\right)$$

in $\tilde{\varphi} = \hat{U} \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$ open and $\tilde{\varphi}$ is bijective onto its image as

$$\left(x^1, \dots, x^n, v^1, \dots, v^n\right) \longmapsto \Big(\underset{\substack{\wedge \\ \varphi^{-1}x,}}{v^i} \frac{\partial}{\partial x^i}\Big|_{\varphi^{-1}x}\Big)$$

is an inverse.

For smooth charts (U, φ), (V, ψ) on M get transition maps

$$\tilde{\psi} \circ \tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \longrightarrow \psi(U \cap V) \times \mathbb{R}^n$$

$\tilde{x}^1, \dots, \tilde{x}^n$ coords of ψ

$$(x, v) = (x^1, \dots, x^n, v^1, \dots, v^n) \longmapsto \Big(\underbrace{\tilde{x}^1(x), \dots, \tilde{x}^n(x)}_{\psi(\varphi^{-1}(x))}, \frac{\partial \tilde{x}^1}{\partial x^j}(x) v^j, \dots, \frac{\partial \tilde{x}^n}{\partial x^j}(x) v^j\Big)$$

which is smooth.

If $\{U_i\}$ is a countable cover of M by smooth coord charts, then $\{\pi^{-1}U_i\}$ is a countable cover of TM by smooth coordinate charts satisfying the hypotheses of the smooth mfld chart lemma.

<u>Prop</u> This makes TM a $2n$-dim'l smooth mfld with $\pi : TM \longrightarrow M$ smooth.

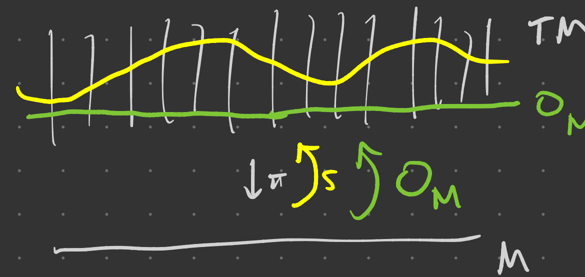
<u>Pf</u> Just need to check that π is smooth. For (U,φ) smooth chart for M, the coord rep'n of π on $(\pi^{-1}U, \tilde{\varphi})$ is $(x,v) \longmapsto x$ which is smooth. $\square$

<u>Note</u>

- For (U,φ) coord chart for M, $\pi^{-1}U \approx U \times \mathbb{R}^n$
- Call the tangent <u>trivial</u> when

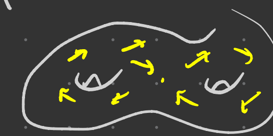
$$\begin{tikzcd} TM \arrow[rr, "\exists \approx"] \arrow[dr, "\pi"'] & & M \times \mathbb{R}^n \arrow[dl, "p_1"] \\ & M & \end{tikzcd}$$

- A section of $\pi : TM \longrightarrow M$ is $s : M \longrightarrow TM$ s.t. $\pi \circ s = \mathrm{id}_M$



i.e. $s(p) \in T_pM$

May also call s a vector field.



- Always have the zero section $0_M : M \longrightarrow TM$, $p \longmapsto (p, 0)$

- A section is nonvanishing if $\mathrm{im}(s) \cap \mathrm{im}\, 0_M = \emptyset$.

- Trivial bundles have nonvanishing sections: fix $v \in \mathbb{R}^n \setminus 0$,

$$s_v : M \longrightarrow M \times \mathbb{R}^n, \quad p \longmapsto (p, v).$$

TPS Why is TS^2 nontrivial?

Hairy ball thm: No nonvanishing cts vector field on S^2

$\implies TS^2 \ncong S^2 \times \mathbb{R}^2$

For $F: M \to N$ smooth, define its global differential

$$dF: TM \to TN$$
$$(p, v) \mapsto (F(p), dF_p(v))$$

Prop If $F: M \to N$ is smooth, then $dF: TM \to TN$ is smooth

and
$$\begin{tikzcd} TM \arrow[r, "dF"] \arrow[d, "\pi_M"'] & TN \arrow[d, "\pi_N"] \\ M \arrow[r, "F"'] & N \end{tikzcd}$$
commutes.

Pf Coordinate rep'n of dF is

$$dF(x,v) = (F(x), \underbrace{JF(x)\cdot v}_{\frac{\partial F^1}{\partial x^i}(x)v^i, \dots, \frac{\partial F^n}{\partial x^i}(x)v^i}).$$

Smooth b/c F is smooth. $\square$

Cor $M \xrightarrow{F} N \xrightarrow{G} P$ smooth

(a) $d(G\circ F) = dG \circ dF$

(b) $d(\mathrm{id}_M) = \mathrm{id}_{TM}$

(c) F a diffeo $\Rightarrow$ dF a diffeo and $(dF)^{-1} = d(F^{-1})$.

Thus we have a functor

$$\begin{array}{ccc} \mathrm{Diff} & \longrightarrow & \mathrm{Bun} \\ M & \longmapsto & TM \xrightarrow{\pi_M} M \\ F\big\downarrow & \longmapsto & dF\big\downarrow \qquad \big\downarrow F \\ N & \longmapsto & TN \xrightarrow[\pi_N]{} N \end{array}$$

category of vector bundles over smooth mflds + bundle maps — see Ch. 10.