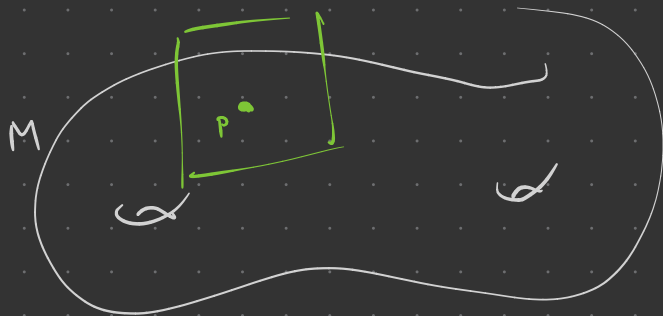
6.II.23

Tangent Vectors

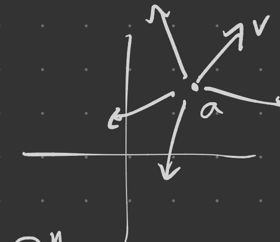


How can we define the tangent space to a point $p \in M$ a smooth manifold?

Desiderata

- Since M has a smooth chart $\varphi : U \longrightarrow \mathbb{R}^n$ in a nbhd of p, T_pM should have the same structure as tangents to $a = \varphi(p) \in \mathbb{R}^n$. Geometrically, these are $\{(a,v) \mid v \in \mathbb{R}^n\} \cong \mathbb{R}^n$.

- What can tangent vectors do?



Given $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ smooth

(i.e. $f \in C^\infty(\mathbb{R}^n)$) and $v|_a := (a,v) \in \{a\} \times \mathbb{R}^n$

we have a directional derivative

$$D_{v|_a} f = (D_v f)(a) = \frac{d}{dt}\Big|_{t=0} f(a+tv)$$

measuring rate of change of f in v direction.

- The function $D_{v|_a} : C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$ is $\mathbb{R}$-linear and satisfies the Leibniz/product rule

$$D_{v|_a}(fg) = f(a)\, D_{v|_a}(g) + D_{v|_a}(f)\, g(a)$$

- So T_pM "should"
  (a) be an n-dim'l real vector space, and
  (b) each $w \in T_pM$ should induce a derivation $C^\infty(M) \longrightarrow \mathbb{R}$ that we can think of as the directional derivative in the w direction.

$\mathbb{R}$-linear map satisfying Leibniz rule: $w(fg) = f(a)\,w(g) + w(f)\,g(a)$

Defn The tangent space to M at $p \in M$ is T_pM, the set of derivations $C^\infty(M) \longrightarrow \mathbb{R}$.

__Reality check__ For this to make sense, need

$$\{a\} \times \mathbb{R}^n \xrightarrow{\cong} T_a \mathbb{R}^n$$
$$v|_a \longmapsto D_{v|_a}$$

i.e. all derivations at a on $\mathbb{R}^n$ should be given by directional derivatives along geometric tangent vectors.

__Lemma__ Suppose $a \in \mathbb{R}^n$, $w \in T_a \mathbb{R}^n$, $f, g \in C^\infty(\mathbb{R}^n)$.

(a) $w(\text{const}) = 0$

(b) If $f(a) = g(a) = 0$, then $w(fg) = 0$.

__Pf__ (a) $w(\underbrace{c_1}_{\text{constant at } 1}) = w(c_1 c_1) = c_1(a)\, w(c_1) + w(c_1)\, c_1(a)$

$= 2 w(c_1) \implies w(c_1) = 0.$

For $c \in \mathbb{R}$, $\text{const}_c = c \cdot c_1$ so $w(\text{const}_c) = c\, w(c_1) = 0$.

(b) $w(fg) = \cancel{f(a)}^{0} w(g) + w(f)\cancel{g(a)}^{0} = 0$. $\square$

Prop For $a \in \mathbb{R}^n$, the map $v|_a \longmapsto D_v|_a$ is an iso $\{a\} \times \mathbb{R}^n \cong T_a\mathbb{R}^n$.

Pf Linear ✓

Injective: Suppose $D_v|_a = 0$. Write $v|_a = \underbrace{v^i e_i|_a}$.

Take $f = x^j : \mathbb{R}^n \to \mathbb{R}$ the j-th coord fn. Then

$$0 = D_{v|_a}(x^j) = v^i \frac{\partial}{\partial x^i}(x^j)\Big|_{x=a} = v^j.$$

Einstein summation! $v^i e_i|_a = \sum_{i=1}^{n} v^i e_i|_a$

True $\forall j$, so $v|_a = (a, 0)$, the 0-vector in $\{a\} \times \mathbb{R}^n$.

Surjective Take $w \in T_a\mathbb{R}^n$ arbitrary. Let $v^i = w(x^i)$.
WTS that $w = D_v|_a$. For $f \in C^\infty(\mathbb{R}^n)$, we have

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)\cdot(x^i - a^i) + \sum_{i,j=1}^{n} \underbrace{(x^i-a^i)(x^j-a^j)\cdot\int_0^1 (1-t)\frac{\partial^2 f}{\partial x^i \partial x^j}(a+t(x-a))\,dt}$$

by Taylor's Thm.

product of smooth fns vanishing at a ! $\Rightarrow w(\sim\sim) = 0$.

Thus $w(f) = w(\cancelto{0}{f(a)}) + \sum_{i=1}^{n} w\left(\frac{\partial f}{\partial x^i}(a)(x^i-a^i)\right) + 0$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)\left(w(x^i) - \cancelto{0}{w(a^i)}\right)$$

$$= \sum \frac{\partial f}{\partial x^i}(a)\, v^i = D_v|_a f.$$

$w : C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$

So $v|_a \longmapsto w = D_{v|a}$, proving surjectivity. $\square$

<u>Cor</u> The derivations $\frac{\partial}{\partial x^i}\Big|_a : f \longmapsto \frac{\partial f}{\partial x^i}(a)$, $i = 1, \dots, n$

form a basis for $T_a\mathbb{R}^n$. $\square$

$T_pM = \{ w : C^\infty(M) \to \mathbb{R} \mid$ w lin, w satisfies Leibniz at $p \}$

We should now feel emboldened to define T_pM as the vector space of derivations (at p) of smooth functions $M \longrightarrow \mathbb{R}$. :)

<u>Note</u> $p \in M$, $v \in T_pM$, $f, g \in C^\infty(M)$ then

- $v(\text{const}) = 0$
- $f(p) = g(p) = 0 \implies v(fg) = 0$.

<u>TPS</u> What is $v(f^2)$

$= v(f \cdot f)$?

$v(f^2) = f(p)\, v(f) + v(f)\, f(p)$

$= 2 f(p)\, v(f)$

∃ other (equivalent!) definitions of T_pM :

- equiv classes of smooth curves through p
- $I := \{f \in C^\infty(M) \mid f(p) = 0\}$, $T_p^*M := I/I^2$ is the cotangent space of M at p. Each $v \in T_pM$ satisfies $v(I^2) = 0$ so induces $T_p^*M \longrightarrow \mathbb{R} \in (T_p^*M)^*$; this correspondence $T_pM \longrightarrow (T_p^*M)^*$ is an iso.

Differential of a smooth map

M, N smooth mflds, $F : M \longrightarrow N$ smooth induces

$$T_pM \xrightarrow{dF_p} T_{F(p)}N$$

$$f \longmapsto f \circ F$$

$$\begin{tikzcd} C^\infty(M) \arrow[d, "v"'] \\ \mathbb{R} \end{tikzcd} \longmapsto \begin{tikzcd} C^\infty(N) \arrow[r] \arrow[dr, "dF_p(v)"'] & C^\infty(M) \arrow[d, "v"] \\ & \mathbb{R} \end{tikzcd}$$

i.e. $dF_p : T_pM \longrightarrow T_{F(p)}(N)$

$$v \longmapsto (f \mapsto v(f\circ F)) =: dF_p(v).$$

This is the differential of F at p.

$dF_p(v) : C^\infty(N) \longrightarrow \mathbb{R}$ linearly ✓

Derivation:

$$\begin{aligned} dF_p(v)(fg) &= v((fg)\circ F) = v((f\circ F)(g\circ F)) \\ &= (f\circ F)(p)\cdot v(g\circ F) + v(f\circ F)\cdot(g\circ F)(p) \\ &= f(F(p))\, dF_p(v)(g) + dF_p(v)(f)\, g(F(p)) \end{aligned}$$ ✓

Properties of Differentials M, N, P smooth mflds w/ or w/o ∂,

$F : M \to N$, $G : N \to P$ smooth, $p \in M$:

(a) $dF_p : T_pM \longrightarrow T_{F(p)}N$ is linear

(b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p$, i.e.,

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[dr, "G \circ F"'] & N \arrow[d, "G"] \\ & P \end{tikzcd} \quad \overset{d}{\rightsquigarrow} \quad \begin{tikzcd} T_pM \arrow[r, "dF_p"] \arrow[dr, "d(G \circ F)_p"'] & T_{F(p)}N \arrow[d, "dG_{F(p)}"] \\ & T_{G(F(p))}P \end{tikzcd}$$

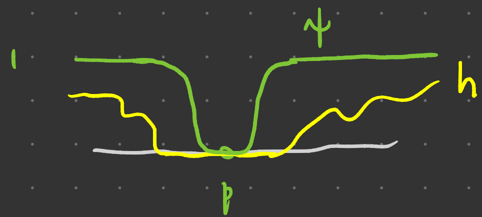
(c) $d(Id_M)_p = Id_{T_pM}$

(d) If F is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$. $\square$

$\mathrm{Diff}_* \longrightarrow \mathrm{Vect}$ $\quad (M,p) \longmapsto T_pM$ is a functor
$\quad F \longmapsto dF_p = T_pF$

<u>Lemma</u> If $f, g \in C^\infty(M)$ satisfy $f|_U = g|_U$ for some nbhd U of p and $v \in T_pM$, then $vf = vg$.

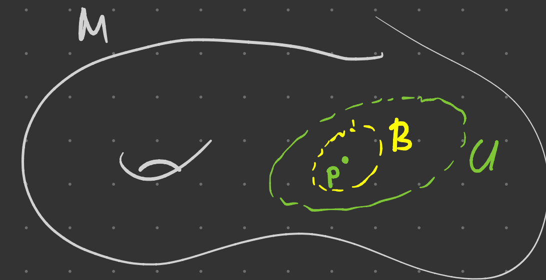
<u>Pf</u> Let $h = f - g$ so that $h \in C^\infty(M)$ with $h|_U = 0$. Let $\psi \in C^\infty(M)$ be a smooth bump fn equal to 1 on supp h and supported on $M \setminus \{p\}$



Then $\psi \cdot h = h$. Since $\psi(p) = h(p) = 0$, get $vh = v(\psi h) = 0$. By linearity of v, $vf = vg$. $\square$

Prop $U \subseteq M$ open, $\iota : U \hookrightarrow M$. $\forall p \in U$, $d\iota_p : T_pU \to T_pM$ is an isomorphism.

Pf Injectivity: Suppose $v \in T_pU$, $d\iota_p(v) = 0$. Take B a nbhd of p with $\bar{B} \subseteq U$.



For $f \in C^\infty(U)$ $\exists \tilde{f} \in C^\infty(M)$ s.t. $\tilde{f}|_{\bar{B}} = f|_{\bar{B}}$. Since f and $\tilde{f}|_U$ $\in C^\infty(U)$ agree on B, know $vf = v(\tilde{f}|_U) = v(\tilde{f} \circ \iota) = d\iota(v)_p \tilde{f} = 0$, Since $f \in C^\infty(U)$ arbitrary, get $v = 0 \Rightarrow d\iota_p$ injective.

Surjectivity Given $w \in T_pM$, define $v : C^\infty(U) \longrightarrow M$

$$f \longmapsto w\underbrace{\tilde{f}}_{\text{any smooth}}$$

Check : v is a derivation ✓

extn to M agreeing with f on $\bar{B}$

For $g \in C^\infty(M)$, $d\iota_p(v)g = v(g \circ \iota) = w(\widetilde{g \circ \iota}) = wg$

lemma

$\square$

<u>Prop</u> If M is an n-dim'l smooth mfld, then $\forall p \in M$, $\dim T_pM = n$.

<u>Pf</u> For $p \in M$, let (U, φ) be a smooth coord chart containing p.

$$\varphi : U \xrightarrow[\text{diffeomorphism}]{} \hat{U} \subseteq \mathbb{R}^n \implies d\varphi_p : \begin{array}{ccc} T_pU & \cong & T_{\varphi(p)}\hat{U} \\ \wr\wr & & \wr\wr \\ T_pM & & T_{\varphi(p)}\mathbb{R}^n \end{array}$$

Know $T_pM \cong T_{\varphi(p)}\mathbb{R}^n \cong \mathbb{R}^n$ of dimn n. $\square$