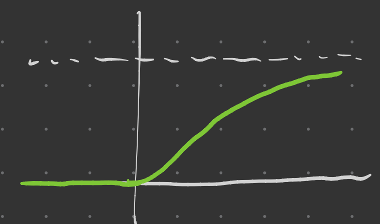
3.II.23

Partitions of Unity

... { Blend together local smooth objects into global ones

Lemma $f:\mathbb{R}\longrightarrow\mathbb{R}$ is smooth

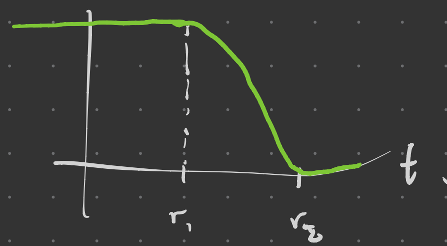
$$t \longmapsto \begin{cases} \exp(-1/t) & t>0 \\ 0 & t\leq 0 \end{cases}$$



Pf **Idea** Just need to check smoothness at $t=0$. Use induction to show $f^{(k)}(t) = p_k(t)\dfrac{\exp(-1/t)}{t^{2k}}$ for p_k polynomial of degree $\leq k$. Then induce to prove $f^{(k)}(0)=0$ $\forall k$. $\square$

Lemma Given $r_1 < r_2 \in \mathbb{R}$, $\exists$ smooth $h: \mathbb{R} \to \mathbb{R}$ s.t.

$h(t) = 1$ for $t \leq r_1$, $0 < h(t) < 1$ for $r_1 < t < r_2$, and $h(t) = 0$ for $t \geq r_2$:



"cutoff function"

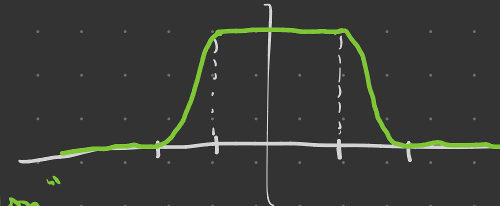
Pf For f as in previous lemma, set $h(t) = \dfrac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}$. $\square$

Lemma For $r_1 < r_2 \in \mathbb{R}$ $\exists$ smoot $H: \mathbb{R}^n \longrightarrow \mathbb{R}$ with

$$H(x) = \begin{cases} 1 & x \in \bar{B}_{r_1}(0) \\ 0 & x \in \mathbb{R}^n \setminus B_{r_2}(0) \end{cases}$$

and $0 < H(x) < 1$ for $x \in B_{r_2}(0) \setminus \bar{B}_{r_1}(0)$

I.e.



"smooth bump function"

Pf Set $H(x) = h(|x|)$ for h as in previous lemma. $\square$

Defn For f a real or vector-valued function on a space M,

$$\operatorname{supp}(f) := \overline{\{p \in M \mid f(p) \neq 0\}}$$

is the **support** of f. If $\operatorname{supp}(f) \subseteq U$ say f is **supported in U**.

If $\operatorname{supp}(f)$ is compact, call f **compactly supported**.

M a space, $\mathfrak{X} = (X_\alpha)_{\alpha\in A}$ open cover of M. A partition of unity subordinate to $\mathfrak{X}$ is an indexed family $(\psi_\alpha : M \to \mathbb{R})_{\alpha \in A}$ such that

(i) $0 \leq \psi_\alpha(x) \leq 1 \quad \forall \alpha \in A,\ x \in M$

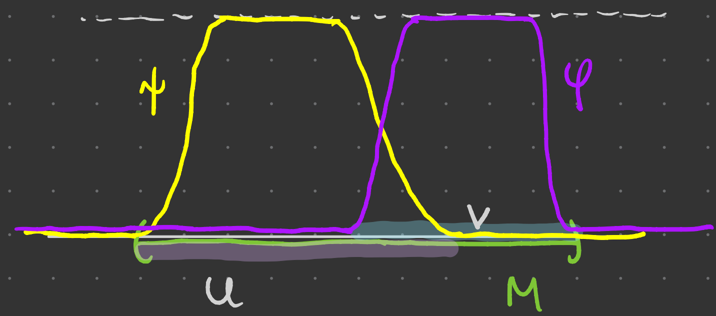
(ii) $\operatorname{supp} \psi_\alpha \subseteq X_\alpha \quad \forall \alpha \in A$

(iii) $\forall x \in M\ \exists$ nbhd U of x s.t. $U \cap \operatorname{supp}\psi_\alpha \neq \emptyset$ for only finitely many α

(iv) $\underbrace{\sum_{\alpha\in A} \psi_\alpha(x)}_{\text{finitely many nonzero terms by (iii)}} = 1 \quad \forall x \in M.$

For M a smooth manifold, a smooth partition of unity is a partition of unity $(\psi_\alpha)_{\alpha\in A}$ with all ψ_α smooth.

E.g.

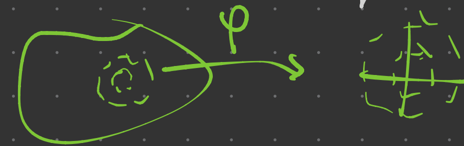


Set $\psi_U = \dfrac{\psi}{\psi+\varphi}$ and $\psi_V = \dfrac{\varphi}{\psi+\varphi}$.

These are smooth with

$$\psi_U + \psi_V = \frac{\psi}{\psi+\varphi} + \frac{\varphi}{\psi+\varphi} = \frac{\psi+\varphi}{\psi+\varphi} = 1.$$

Thm Suppose M is a smooth mfld w/ or w/o ∂, and $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ is any indexed open cover of M. Then $\exists$ smooth part'n of unity on M subordinate to $\mathcal{X}$.

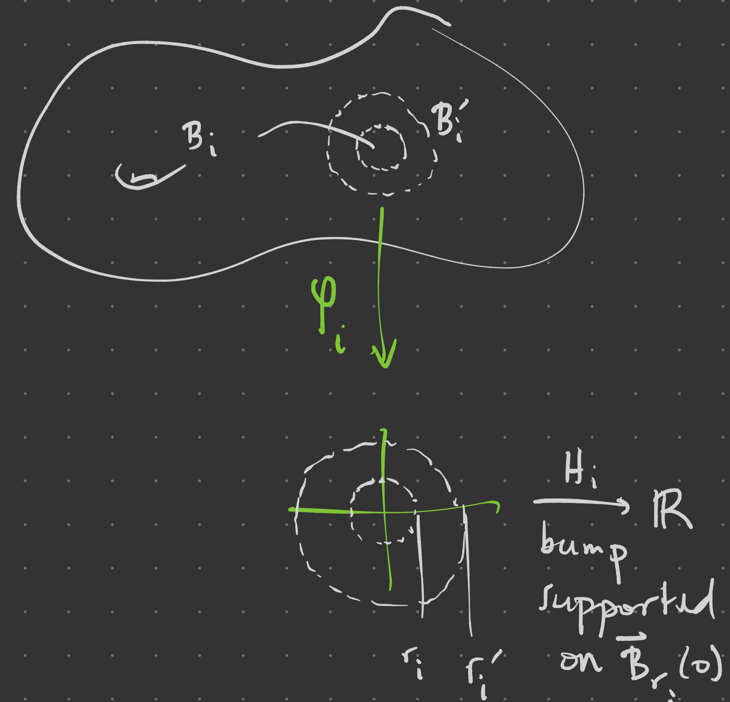


Pf Each X_α is a smooth mfld so has a basis $\mathcal{B}_\alpha$ of reg coord balls. Thus $\mathcal{B} = \bigcup_{\alpha \in A} \mathcal{B}_\alpha$ is a basis for M. By paracompactness theorem, $\mathcal{X}$ has a countable locally finite refinement $\{B_i\}$ consisting of elts of $\mathcal{B}$. $\{\bar{B}_i\}$ is also locally finite (exc).

For B_i a reg coord ball of X_α, take coord ball $B_i' \subseteq X_\alpha$ s.t. $\bar{B}_i \subseteq B_i' \xrightarrow[\varphi_i \text{ smooth}]{} \mathbb{R}^n$, $\varphi_i(\bar{B}_i) = \bar{B}_{r_i}(0)$ and $\varphi_i(B_i') = B_{r_i'}(0)$ for some $r_i < r_i'$.

For each i, define

$$f_i : M \longrightarrow \mathbb{R}$$
$$x \longmapsto \begin{cases} H_i \circ \varphi_i & x \in B_i' \\ 0 & x \in M \setminus B_i' \end{cases}$$



for H_i as in previous lemma; have $\operatorname{supp}(f_i) = \overline{B}_i$

Define $f : M \longrightarrow \mathbb{R}$ by $f(x) = \sum f_i(x)$ (well-defined by local finiteness of $\{\overline{B}_i\}$). Since $\{B_i\}$ covers, $f(x) > 0 \ \forall x \in M$.

Define $g_i : M \longrightarrow \mathbb{R}$, $x \longmapsto \dfrac{f_i(x)}{f(x)}$. Then $0 \le g_i \le 1$ and $\sum g_i = 1$.

Last step: re-index fns by A. For each i, choose $a(i) \in A$ s.t. $B_i' \subseteq X_{a(i)}$. For $\alpha \in A$, define $\psi_\alpha : M \longrightarrow \mathbb{R}$

$$x \longmapsto \sum_{\substack{i \text{ s.t.} \\ a(i) = \alpha}} g_i(x)$$

$(0$ if $\nexists\, i$ s.t. $a(i) = \alpha)$

Have $\operatorname{supp} \psi_\alpha = \overline{\bigcup_{a(i)=\alpha} B_i} = \bigcup_{a(i)=\alpha} \bar{B}_i \subseteq X_\alpha$

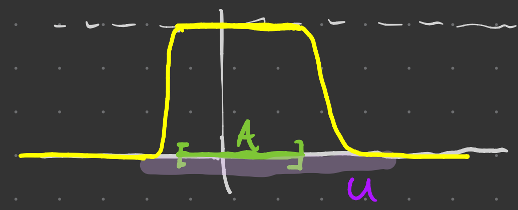
and $\sum_{\alpha \in A} \psi_\alpha$ is a partition of unity subordinate to $\mathcal{X}$. $\square$

Applications

$A \subseteq M$ closed

$U \subseteq M$ open

$A \subseteq U \subseteq M$

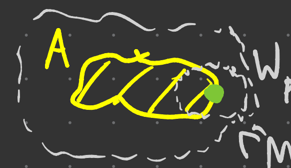


$\psi : M \longrightarrow \mathbb{R}$ s.t. $0 \leq \psi \leq 1$, $\psi|_A = 1$, $\operatorname{supp} \psi \subseteq U$ is called a bump function for A supported in U.

Prop M a smooth mfld w/ or w/o ∂. For any closed $A \subseteq M$, open $U \subseteq M$ containing A $\exists$ smooth bump function for A supported in U.

Pf Let $U_0 = U$, $U_1 = M \setminus A$, $\{\psi_0, \psi_1\}$ a smooth partition of unity subordinate to $\{U_0, U_1\}$. Since $\psi_1 = 0$ on A, have $\psi_0 = \psi_0 + \psi_1 = 1$ on A, so ψ_0 is a smooth bump for A supp in U. $\square$

Lemma M a smooth mfld w/ or w/o ∂, $A \subseteq M$ closed, $f : A \to \mathbb{R}^k$ smooth. For any $A \subseteq U \subseteq M$ open, $\exists$ smooth $\tilde{f} : M \to \mathbb{R}^k$ s.t. $\tilde{f}|_A = f$ and $\operatorname{supp} \tilde{f} \subseteq U$.



Pf For $p \in A$ choose nbhd W_p of p and smooth $\tilde{f}_p : W_p \to \mathbb{R}^k$ s.t. $\tilde{f}_p|_{W_p \cap A} = f|_{W_p \cap A}$. WLOG, $W_p \subseteq U$. Then

$\{W_p \mid p \in A\} \cup \{M \setminus A\}$ is an open cover of M. Take

$\{\psi_p \mid p \in A\} \cup \{\psi_0\}$ smooth part'n of unity subordinate to this cover w/ $\operatorname{supp} \psi_p \subseteq W_p$, $\operatorname{supp} \psi_0 \subseteq M \setminus A$.

For each $p \in A$, $\psi_p \tilde{f}_p$ is smooth on W_p w/ smooth extn to all of M — 0 on $M \setminus \operatorname{supp} \psi_p$.

Define $\tilde{f} : M \longrightarrow \mathbb{R}^k$ smooth!

$$x \longmapsto \sum_{p \in A} \psi_p(x) \tilde{f}_p(x).$$

For $x \in A$, $$\tilde{f}(x) = \sum_{p \in A} \psi_p(x) \underbrace{f(x)}_{\text{b/c } \tilde{f}_p \text{ extends } f \text{ on } W_p} = \underbrace{\Big(\underbrace{\psi_0(x)}_{0 \text{ on } A} + \sum_{p \in A} \psi_p(x) \Big)}_{1} f(x) = f(x)$$

Thus $\tilde{f}$ extends f, $\operatorname{supp} \tilde{f} = \overline{\bigcup_{p \in A} \operatorname{supp} \psi_p} = \bigcup \operatorname{supp} \psi_p \subseteq U$. $\square$

- Codomain can't be an arbitrary mfld ($id : S^1 \to S^1$ does not extend to $\mathbb{R}^2$).

$$\begin{tikzcd} \pi_1 S^1 \arrow[rr, "\cong"] \arrow[d] & & \pi_1 S^1 \\ \pi_1 \mathbb{R}^2 \arrow[urr] & & \end{tikzcd}$$

- A must be closed.
- Extension fails for real-analytic functions.

Partitions of unity tree

- smooth exhaustion functions:
  $f : M \to \mathbb{R}$ smooth with sublevel sets $f^{-1}(-\infty, c]$ compact $\forall c \in \mathbb{R}$
- level sets of smooth fns realize all closed subsets of M:
  $\forall K \subseteq M$ closed $\exists$ smooth $f : M \to \mathbb{R}_{\geq 0}$ with $f^{-1}\{0\} = K$.

∘ ∘ ∘ { f gives height of
pts on M