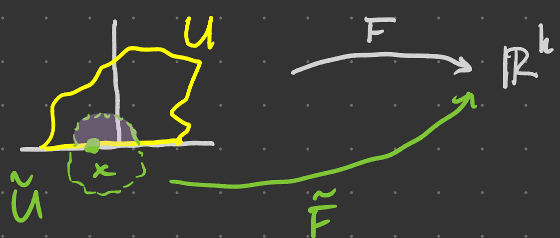
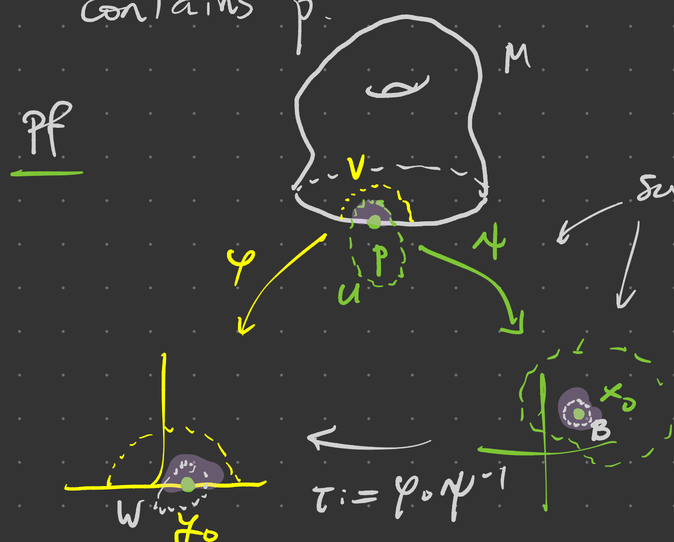
Manifolds w/ Boundary

For $\mathcal{U} \subseteq \mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$, $F : \mathcal{U} \to \mathbb{R}^k$ is smooth if $\forall x \in \mathcal{U}$ $\exists\, \tilde{\mathcal{U}} \subseteq \mathbb{R}^n$ open containing x and smooth $\tilde{F} : \tilde{\mathcal{U}} \longrightarrow \mathbb{R}^k$ s.t. $\tilde{F}|_{\mathcal{U} \cap \tilde{\mathcal{U}}} = F|_{\mathcal{U} \cap \tilde{\mathcal{U}}}$.



For M a top'l mfld with boundary, a smooth structure for M is a maximal smooth atlas for M (w/ above notion of smoothness on ∂M).

Thm (Smooth Invariance of the Boundary) Suppose M is a smooth mfld w/ boundary and $p \in M$. If there is a smooth chart (U, φ) for M s.t. $\varphi(U) \subseteq \mathbb{H}^n$ and $\varphi(p) \in \partial\mathbb{H}^n$, then the same is true for every smooth chart whose domain contains p.



Take W a nbhd of y_0 and smooth $\eta : W \to \mathbb{R}^n$ agreeing with τ^{-1} on $W \cap \varphi(U \cap V)$.

Take B open ball in $\varphi(U\cap V)$ containing x_0. Then τ is smooth on B in the usual sense. WLOG, $B \subseteq \tau^{-1} W$.

Then $\eta \circ \tau|_B = \tau^{-1} \circ \tau|_B = id_B$ so by chain rule,

$$D\eta(\tau(x)) \circ D\tau(x) = Id_{\mathbb{R}^n} \quad \forall x \in B$$

$\Longrightarrow D\tau(x)$ nonsingular $\Longrightarrow \tau$ is open

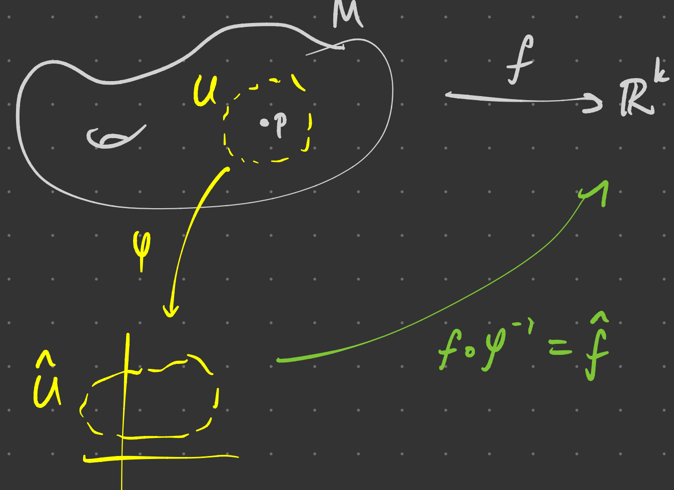
$\Longrightarrow \tau(B)$ open in $\mathbb{R}^n$, contains y_0, and is contained φV

This contradicts $\varphi V \subseteq \mathbb{H}^n$, $\varphi(p) \in \partial\mathbb{H}^n$. ⨳ $\square$

Smooth Functions and Smooth Maps

M smooth mfld, $f: M \to \mathbb{R}^k$ is smooth if $\forall p \in M$ $\exists$ smooth chart (U, φ) s.t. $f \circ \varphi^{-1}$ is smooth on $\hat{U}$.

$$\begin{matrix} U \\ \cup\!\!\!\!| \\ p \end{matrix}$$



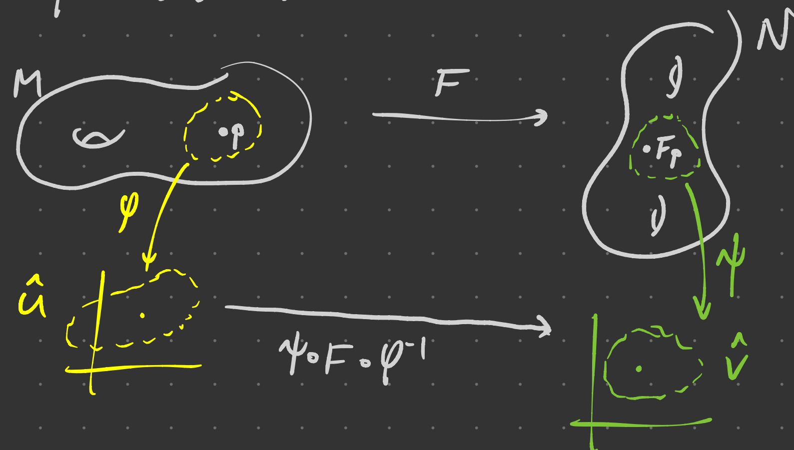
Write $C^\infty(M)$ for the set of smooth functions $M \to \mathbb{R}$.
It's an $\mathbb{R}$-algebra under pointwise add'n, mult'n.

Prop If $f: M \longrightarrow \mathbb{R}^k$ is smooth, then $f \circ \varphi^{-1}$ is smooth $\forall$ smooth chart (U, φ). $\square$

A function $F: M \longrightarrow N$ between smooth mflds is smooth

if $\forall_{p} \in M$ $\exists$ smooth charts $\underset{\substack{\cup \\ p}}{(U, \varphi)}$ on M, $\underset{\substack{\cup \\ FU}}{(V, \psi)}$ on N

s.t. $\psi \circ F \circ \varphi^{-1}$ is smooth on $\hat{U}$:



Prop Every smooth map is cts

homeo

Pf $F|_U = \psi^{-1} \circ (\psi \circ F \circ \varphi^{-1}) \circ \varphi$ is smooth (in the Euclidean sense) hence cts. Thus F is cts at a nbhd of each pt of M hence cts. $\square$

Fact $F: M \longrightarrow N$ is smooth iff

- $\forall p \in M$ $\exists$ smooth charts $(\underset{p}{\underset{\Psi}{U}}, \varphi)$, $(\underset{F(p)}{\underset{\Psi}{V}}, \psi)$ s.t. $U \cap F^{-1}V \subseteq M$ is open and $\psi \circ F \circ \varphi^{-1}$ is smooth $\varphi(U \cap F^{-1}V) \longrightarrow \psi(V)$;

iff

- F is cts and $\exists$ smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$, $\{(V_\beta, \psi_\beta)\}$ for M, N s.t. for each α, β, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is smooth.

iff • $\forall p \in M$ $\exists$ nbhd U of p s.t. $F|_U$ is smooth

<u>Upshot</u> Smooth maps on an open cover that agree on overlaps can be "glued" to give a unique smooth map restricting to the original maps.

<u>Fact</u> If $F: M \longrightarrow N$ is smooth, then every coordinate representation $\hat{F} = \psi \circ F \circ \varphi^{-1}$ is smooth.

<u>Prop</u> M, N, P smooth mflds w/ or w/o ∂.

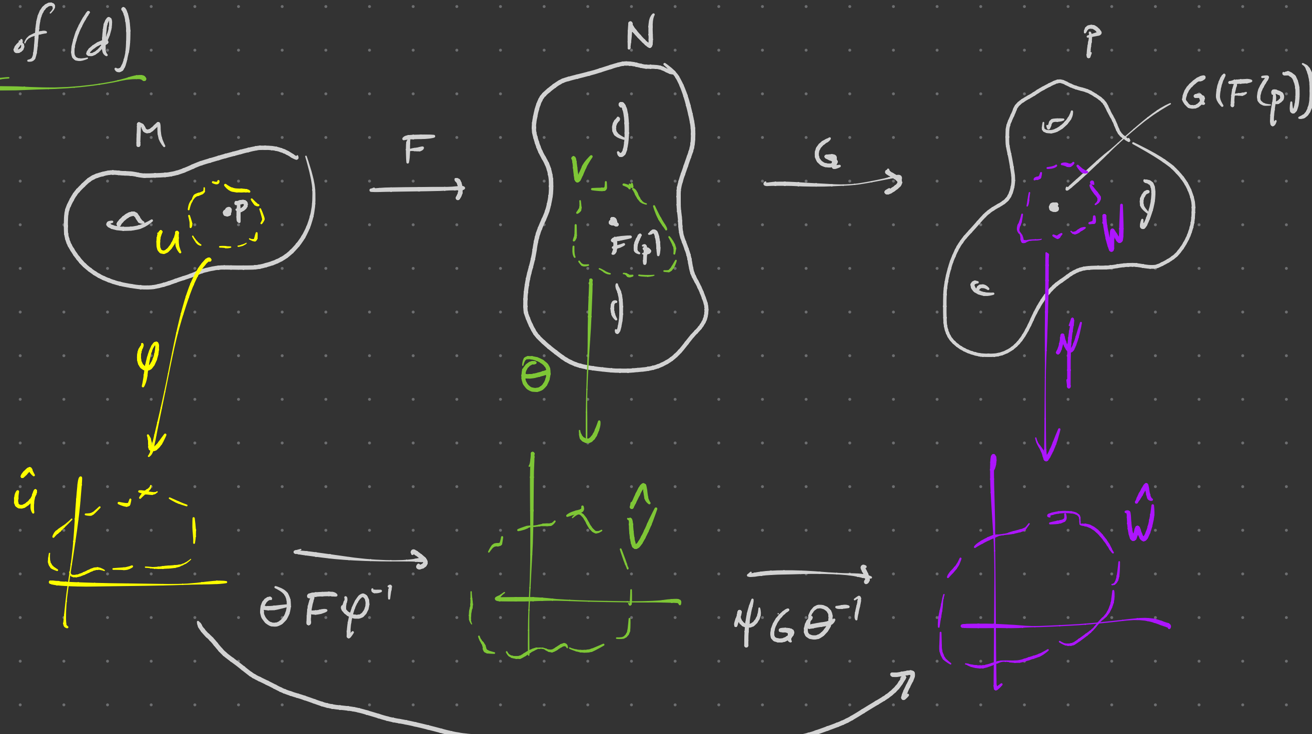
(a) Every constant map $c: M \longrightarrow N$ is smooth.

(b) $id_M: M \longrightarrow M$ is smooth.

(c) $U \subseteq M$ an open submfld w/ or w/o ∂, then $U \hookrightarrow M$ is smooth.

(d) If $F: M \to N$ & $G: N \to P$ are smooth, then $G \circ F: M \to P$ is smooth.

Pf of (d)



$$(\psi G \theta^{-1})(\theta F \varphi^{-1}) = \psi (G \circ F) \varphi^{-1}$$

smooth + smooth $\Longrightarrow$ smooth! $\square$

Note This means we have a category **Diff** w/ objects smooth mflds and $\mathrm{Diff}(M,N) := \{F : M \to N \text{ smooth}\}$.

Prop Products of smooth maps are smooth. $\square$

E.g.

$x^i \neq 0$

$$\begin{tikzcd} S^n \arrow[r, hook] \arrow[dr] & \mathbb{R}^{n+1}\setminus\{0\} \supseteq V_i \arrow[d] \arrow[dr] \\ & \mathbb{RP}^n \supseteq U_i \arrow[r, "\varphi_i"'] & \mathbb{R}^n \end{tikzcd}$$

$$\begin{tikzcd} (x^0, \dots, x^n) \arrow[d, mapsto] \arrow[dr, mapsto] & \\ {[x^0, \dots, x^n]} \arrow[r, mapsto] & \frac{1}{x^i}(x^0, \dots, \widehat{x^i}, \dots, x^n) \end{tikzcd}$$

$\{[x^0, \dots, x^n] \mid x^i \neq 0\}$

$\Longrightarrow S^n \to \mathbb{RP}^n$ smooth.

An isomorphism $F: M \to N$ in Diff is called a diffeomorphism;

it's a smooth map with smooth 2-sided inverse.

If a diffeo $F: M \to N$ exist, call M, N diffeomorphic and

write $M \simeq N$. $\cong$ $\simeq$ $\sim$ $\approx$ \approx

E.g.

- Consider $\mathbb{R}$ with its standard smooth structure

  and $\tilde{\mathbb{R}} = \mathbb{R}$ + smooth structure induced by

$$\left\{ \left(\mathbb{R}, \begin{array}{c} \psi : \mathbb{R} \to \mathbb{R} \\ x \mapsto x^3 \end{array} \right) \right\}$$

  Define $\begin{array}{rcl} F : \mathbb{R} & \longrightarrow & \tilde{\mathbb{R}} \\ x & \longmapsto & x^{1/3} \end{array}$. This has coordinate

  representation $\hat{F} = \psi \circ F \circ id_{\mathbb{R}}^{-1} = id_{\mathbb{R}}$ which is smooth

  and $F^{-1}: x \mapsto x^3$ has coord rep

$\widehat{F}^{-1} = id_{\mathbb{R}} \circ F^{-1} \circ \psi^{-1} = id_{\mathbb{R}}$ which is smooth.

Thus $\mathbb{R} \approx \tilde{\mathbb{R}}$. In fact, every smooth structure on $\mathbb{R}$ is diffeomorphic to the standard one!

(See Prob 15-13.)

- $\mathbb{B}^n \approx \mathbb{R}^n$

$$x \longmapsto \frac{x}{\sqrt{1-|x|^2}}$$

$$\frac{y}{\sqrt{1+|y|^2}} \longleftarrow y$$

- There are smooth structures on $\mathbb{R}^4$ not diffeomorphic to the standard smooth structure.

- S^7 carries exactly 15 diffeo classes of smooth structures.

<u>Thm</u> (Diffeomorphism invariance of dimn & boundary)

If $F: M \approx N$, then $\dim M = \dim N$ and $F(\partial M) = \partial N$, $F|_{M^\circ}: M^\circ \approx N^\circ$.

Pf of dim invariance

$$
\begin{tikzcd}
F : M^m \arrow[r, "\approx"] & N^n \\
p \in U \arrow[u, phantom, sloped, "\subseteq"] \arrow[d, "\varphi"'] & V \ni F(p) \arrow[u, phantom, sloped, "\subseteq"] \arrow[d, "\psi"] \\
\hat{U} \arrow[r, "\hat{F}"'] & \hat{V}
\end{tikzcd}
$$

φ, ψ smooth coord charts

Then $\hat{F}$ is a (Euclidean) diffeo from an open subset of $\mathbb{R}^m$ to an open subset of $\mathbb{R}^n$.

Prop C.4 Then $m=n$ and $\forall a \in \hat{U}$, $D\hat{F}(a)$ is invertible with

$$D\hat{F}(a)^{-1} = D(\hat{F}^{-1})(\hat{F}(a)).$$

Indeed, $\hat{F}^{-1} \circ \hat{F} = id_{\hat{U}} \underset{\text{chain rule}}{\Longrightarrow} id_{\mathbb{R}^m} = D(\hat{F}^{-1})(\hat{F}(a)) \circ D\hat{F}(a)$.

Similarly, $\mathrm{id}_{\mathbb{R}^n} = D\hat{F}(a) \circ D(\hat{F}^{-1})(\hat{F}(a))$.

Thus $D\hat{F}(a)$ is a linear isomorphism $\Longrightarrow m = n$. $\square$

Any time you can take advantage of linear algebra, do!

$$(r, \theta) \longmapsto (r\cos\theta, r\sin\theta)$$

$$z \longmapsto z^2$$

$$(x, y) \longmapsto (x^2 - y^2, -2xy)$$

$$J = \begin{pmatrix} 2x & -2y \\ -2y & -2x \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{singular!}$$