30.I.23

# Smooth manifolds

Recall that a topological manifold $M$ of dimn $n$ is a

- Hausdorff,
- second countable,
- locally Euclidean of dimn $n$

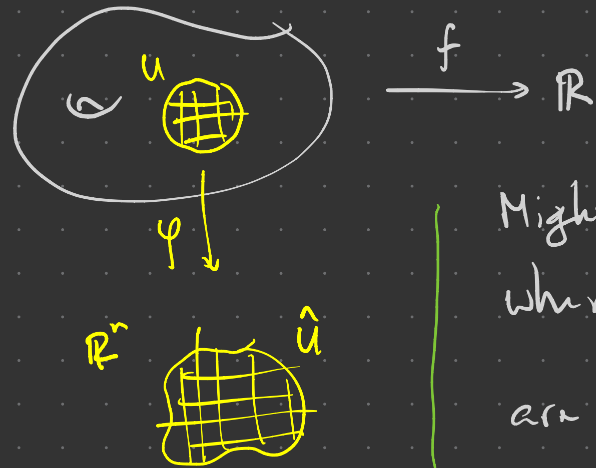
space. In particular, $\forall p \in M$ $\exists$ $\varphi$ :
$$\begin{array}{ccc} p & & \\ \cap & & \\ U & \xrightarrow{\cong} & \hat{U} \\ \text{open} \cap & & \cap \text{ open} \\ M & & \mathbb{R}^n \end{array}.$$

Call $(U, \varphi)$ a coordinate chart on $M$.

The map $\varphi$ is a local coordinate map with component functions $(x^1, \ldots, x^n)$, i.e. $\varphi(m) = (x^1(m), x^2(m), \ldots, x^n(m)) \in \hat{U} \subseteq \mathbb{R}^n$.

**Q** Can we do calculus on topological manifolds?

**A** No!



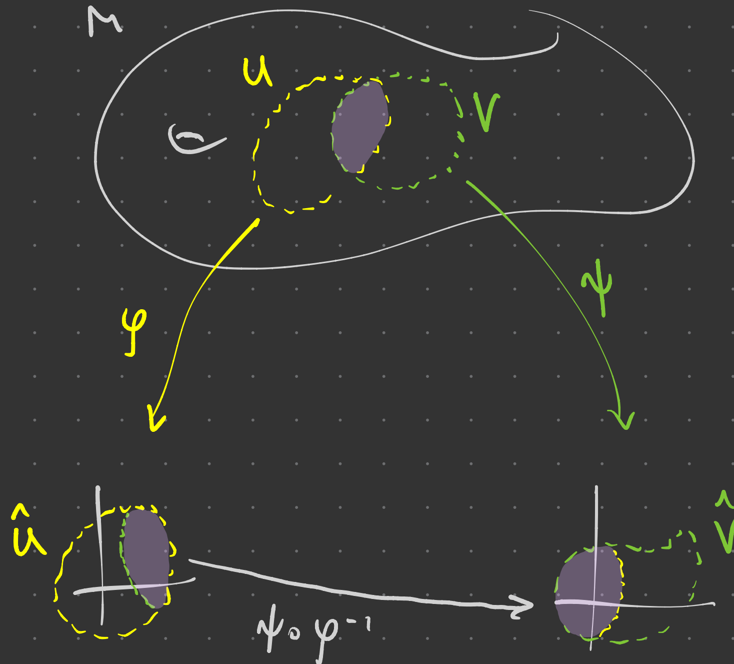

Might like to say $f$ is differentiable when all maps $\hat{U} \xrightarrow[\varphi^{-1}]{} U \subseteq M \to \mathbb{R}$ are differentiable.

**Issue** Can compose $\varphi$ w/ homeomorphisms that destroy this property!

**Solution**

(1) Make charts part of the structure

(2) Demand smooth compatibility of charts.




Charts $(U,\varphi)$, $(V,\psi)$ are smoothly compatible when the transition map

$\psi\circ\varphi^{-1}:\varphi(U\cap V)\to\psi(U\cap V)$

is a diffeomorphism: smooth bijection with smooth inverse.

each component function has cts partial derivatives of all orders

equivalently, $\psi\circ\varphi^{-1}$ is smooth with nonsingular Jacobian at each point.

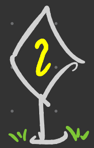
An *atlas* for $M$ is a collection of charts whose domains cover $M$.

An atlas $\mathcal{A}$ is a *smooth atlas* if any two charts in $\mathcal{A}$ are smoothly compatible.

*Note* Suffices to check that $\psi \circ \varphi^{-1}$ is smooth $\forall (U, \varphi), (V, \psi) \in \mathcal{A}$.

A smooth atlas $\mathcal{A}$ on $M$ is *maximal* when $\mathcal{A} \subseteq \mathcal{A}'$ for some other smooth atlas $\mathcal{A}'$ on $M$ $\Longrightarrow$ $\mathcal{A} = \mathcal{A}'$.

*Defn* A *smooth structure* on a topological manifold $M$ is a maximal smooth atlas. A *smooth manifold* is a pair $(M, \mathcal{A})$ with $M$ a top'l mfld and $\mathcal{A}$ a smooth structure on $M$.



- Some top'l mflds have $>1$ smooth structure.
- Some top'l mflds have no smooth structures.

**Prop** Let $M$ be a top'l mfld.

(a) Every smooth atlas $\mathcal{A}$ for $M$ is contained in a unique maximal smooth atlas — the smooth structure determined by $\mathcal{A}$.

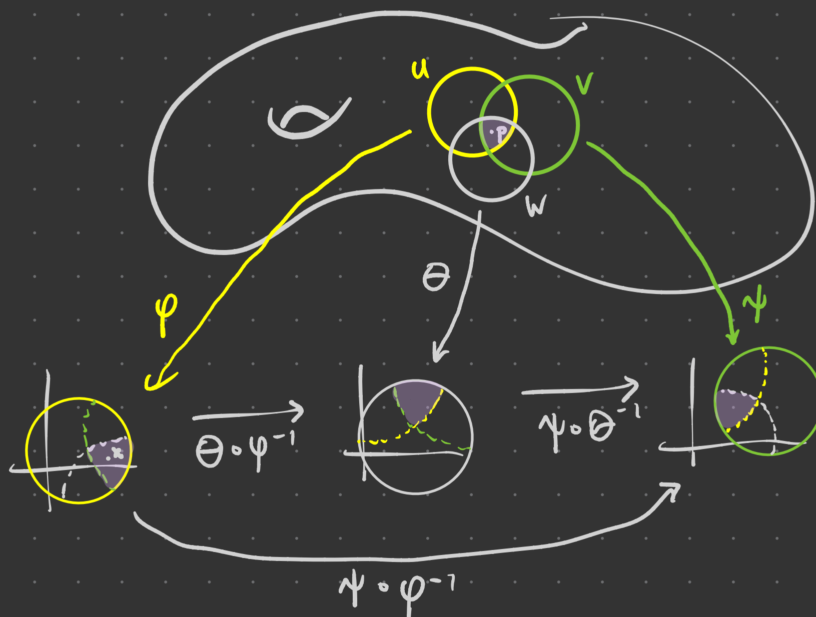
Exc (b) Two smooth atlases for $M$ determine the same smooth structure iff their union is a smooth atlas.

Pf of (a) Let $\bar{\mathcal{A}} = \{$charts smoothly compatible w/ every chart in $\mathcal{A}\}$

WTS $\bar{\mathcal{A}}$ is a smooth atlas. This suffices as $\mathcal{A} \subseteq \mathcal{B}$ = a max'l smooth atlas $\Rightarrow \mathcal{B} \subseteq \bar{\mathcal{A}} \Rightarrow \mathcal{B} = \bar{\mathcal{A}}$.

For $(U,\varphi), (V,\psi) \in \bar{\mathcal{A}}$, must prove $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is smooth. Take $x = \varphi(p) \in \varphi(U \cap V)$ arbitrary. Since $\mathcal{A}$ is an atlas, $\exists\, (W, \Theta) \in \mathcal{A}$ s.t. $p \in W$. By defn of $\bar{\mathcal{A}}$,

$\Theta \circ \varphi^{-1}$ and $\psi \circ \Theta^{-1}$ are smooth.

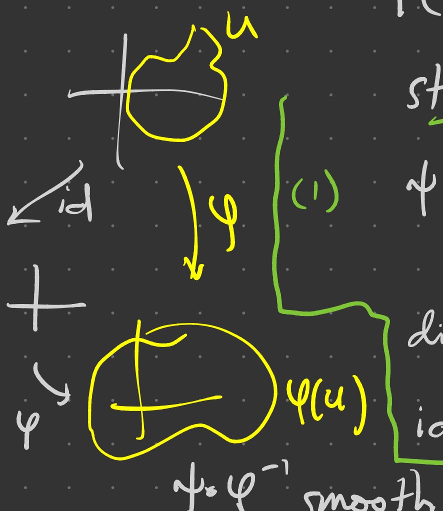


Since $p \in U \cap V \cap W$, $\psi\varphi^{-1} = (\psi\theta^{-1})(\theta\varphi^{-1})$ is smooth on a nbhd of $x$.

Thus $\psi \circ \varphi^{-1}$ is smooth on a nbhd of each point of $\varphi(U \cap V)$

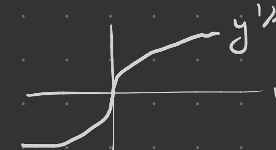
$\Rightarrow \bar{A}$ is a smooth atlas. $\square$

**E.g.** (0) The smooth structure determined by the atlas $\{(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})\}$ on $\mathbb{R}^n$ is the standard smooth structure on $\mathbb{R}^n$.




(1) $\psi : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto x^3$ defines a smooth structure on $\mathbb{R}$ distinct from the standard smooth structure:

$\mathrm{id} \circ \psi^{-1} : y \longmapsto y^{1/3}$ not smooth at $0$



(2) Real $m \times n$ matrices $\mathbb{R}^{m \times n}$ have a smooth structure determined by

$$\begin{pmatrix} x^{11} & x^{12} & \cdots & x^{1n} \\ x^{21} & x^{22} & \cdots & x^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x^{m1} & x^{m2} & \cdots & x^{mn} \end{pmatrix} \longmapsto (x^{11}, \dots, x^{mn}) \in \mathbb{R}^{mn}$$

Similarly get a smooth structure on $\mathbb{C}^{m \times n} \cong (\mathbb{R}^2)^{m \times n}$.

(3) For $U \subseteq M$ open, may restrict charts on $M$ to get a smooth structure on $U$.

(4) The general linear group $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus 0)$ $\subseteq \mathbb{R}^{n \times n}$ is open (b/c $\det : \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}$ is cts) so has an induced smooth structure.

(5) $U \subseteq \mathbb{R}^n$ open, $f : U \longrightarrow \mathbb{R}^k$ smooth then

$\Gamma(f) := \{ (u, f(u)) \mid u \in U \} \subseteq U \times \mathbb{R}^k$ is a top'l mfld

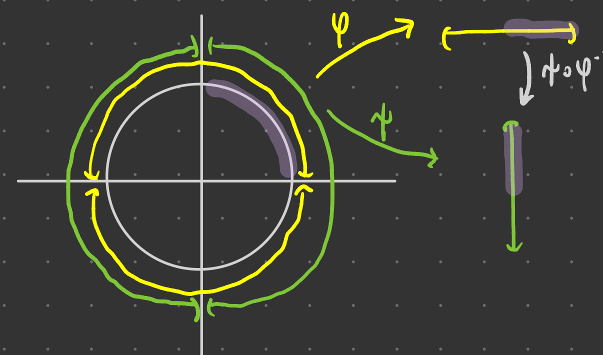
and smooth structure is induced by projection.

$$\begin{tikzcd} \Gamma(f) \subseteq U \times \mathbb{R}^k \arrow[dr, "\varphi"'] \arrow[d, "\pi"] & (u, x) \arrow[d, mapsto] \\ U & u \end{tikzcd}$$

(6) $S^n \subseteq \mathbb{R}^{n+1}$ has a smooth atlas given by hemispheres

and projection :

$\left( \frac{\partial}{\partial x} (x^2 + y^2) = 2x, \ 2y \right)$




$$\begin{array}{c} x \\ \downarrow \\ (x, \sqrt{1-x^2}) \\ \downarrow \\ \sqrt{1-x^2} \end{array}$$

smooth on $(0,1)$ ✓

(7) (Level sets) $U \subseteq \mathbb{R}^n$ open, $\Phi : U \longrightarrow \mathbb{R}$ smooth.

For any $c \in \mathbb{R}$, $\Phi^{-1}\{c\}$ is a level set of $\Phi$. Fix $c \in \mathbb{R}$, set $M := \Phi^{-1}\{c\}$ and suppose $D\Phi(a) = \left(\frac{\partial \Phi}{\partial x^1}(a), \dots, \frac{\partial \Phi}{\partial x^n}(a)\right) \neq 0$ for each $a \in M$. Thus $\exists\, i$ s.t. $\frac{\partial \Phi}{\partial x^i}(a) \neq 0$ for each $a$, so by the implicit function theorem $\exists$ open $U_0 \subseteq U$ containing $a$ s.t. $M \cap U_0$ is the graph of an eqn of the form

$$x^i = f(x^1, \dots, \underbrace{\hat{x}^i}_{\text{omit } x^i \text{ input}}, \dots, x^n).$$

Use the $M \cap U_0$ sets w/ proj'n onto $\hat{U}_0 \subseteq \mathbb{R}^{n-1}$ to give $M$ a smooth structure. (Just like $S^n$!)

(8) ($\mathbb{RP}^n$) The standard charts for $\mathbb{RP}^n$ are

$$\left( U_i = \{ [x^0, \dots, x^n] \mid x^i \neq 0 \},\ \begin{aligned} \varphi_i : U_i &\longrightarrow \mathbb{R}^n \\ [x] &\longmapsto \tfrac{1}{x^i}(x^0, \dots, \hat{x^i}, \dots, x^n) \end{aligned} \right).$$

On $U_i \cap U_j$ we have

$$\varphi_j \circ \varphi_i^{-1}(x^1, \dots, x^n) = \frac{1}{x^j}(x^1, \dots, x^{j-1}, x^{j+1}, \dots, x^{i-1}, 1, x^i, \dots, x^n)$$

which is a diffeo $\varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$.

$$\begin{array}{ccc} \varphi_i(U_i \cap U_j) & \longrightarrow & \varphi_j(U_i \cap U_j) \\ \| & & \| \\ \mathbb{R}^n \setminus \{x^j = 0\} & & \mathbb{R}^n \setminus \{x^i = 0\} \end{array}$$

(9) Products of smooth mflds have smooth structures induced by products of charts.

## (10) (Grassmannians)

Herrmann Grassmann (1809–1877)

**Lemma** (Smooth mfld chart lemma) $M$ a set, $\{U_\alpha\}_\alpha$ subsets of $M$ with $\varphi_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$ s.t.

(i) $\forall \alpha$, $\varphi_\alpha$ is a bij'n onto $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ open

(ii) $\forall \alpha, \beta$, $\varphi_\alpha(U_\alpha \cap U_\beta)$, $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$

(iii) If $U_\alpha \cap U_\beta \neq \emptyset$, then $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth

(iv) Countably many of the $U_\alpha$ cover $M$

(v) For $p \neq q \in M$, either $\exists U_\alpha \ni p, q$ or $\exists U_\alpha, U_\beta$ disjoint with $p \in U_\alpha$, $q \in U_\beta$.

Then $M$ is a unique smooth mfld w/ charts $\varphi_\alpha$. Pf p.22 $\square$

Let $V$ be an $n$-dim'l real vector space. For $0 \leq k \leq n$, let

$$G_k(V) := \{ P \leq V \mid P \text{ } k\text{-dim'l linear subspace of } V \}.$$

Note $G_{k,n} := G_k(\mathbb{R}^n)$, $G_{1,n+1} = \mathbb{RP}^n$.

Claim We can give $G_k(V)$ the structure of a $k(n-k)$-dim'l smooth mfld.

For $Q \leq V$ of dimn $n-k$, define $\mathcal{U}_Q := \{ P \in G_k(V) \mid P \cap Q = 0 \}$.

These will form our coordinate nbhds.

If $V = \underset{\text{dim } k}{P} \oplus \underset{n-k}{Q}$, the graph of any linear map $X : P \longrightarrow Q$

can be identified with $\Gamma(X) = \{ v + Xv \mid v \in P \} \in G_k(V)$

and we have $\Gamma(X) \cap Q = 0$.

Conversely, if $S \in G_k(V)$ has $S \cap Q = 0$, then

$\pi_P|_S : S \xrightarrow{\cong} P \implies X = (\pi_Q|_S) \circ (\pi_P|_S)^{-1} : P \to Q$ linear

with $\Gamma(X) = S$.

Write $\mathcal{L}(P, Q)$ for the vector space of linear maps $P \to Q$.

Now have $\Gamma : \mathcal{L}(P,Q) \longrightarrow U_Q$ which is bijective.

Define $\varphi = \Gamma^{-1} : U_Q \xrightarrow[\text{bij}]{} \mathcal{L}(P, Q) \cong \mathbb{R}^{(n-k)\times k} \cong \mathbb{R}^{k(n-k)}$,

verifying (i) of lemma.

More work for other properties — see pp. 23-24.