27. I. 23

Homology of CW complexes

$\partial_* \neq (f)_*$

Let's attach an n-cell, $n \geq 2$, to a space X:

$$\begin{array}{ccc} \partial D & \xrightarrow{\varphi} & X \\ \downarrow & \ulcorner & \downarrow \\ D & \longrightarrow & Y \end{array}$$

Let $q : X \sqcup D \longrightarrow Y$ be the quotient map gluing D to X.

Set $U = q(D^\circ) \cong D^\circ$, $V = q(X \sqcup D \setminus 0) \simeq X$, $U \cap V \cong D^\circ \setminus 0 \simeq \partial D$

M-V for $Y = U \cup V$ then looks like

$$
\begin{tikzcd}
H_p(U\cap V) \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & H_p(\cancel{U}) \oplus H_p(V) \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & H_p(Y) \arrow[r] & H_{p-1}(U\cap V) \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & H_{p-1}(\cancel{U}) \oplus H_{p-1}(V) \arrow[d, phantom, "\cong" rotate=-90] \\
H_p(\partial D) \arrow[r, "\varphi_*"'] & H_p(X) & & H_{p-1}(\partial D) \arrow[r, "\varphi_*"'] \arrow[d, phantom, "\cong" rotate=-90] & H_{p-1}(X) \\
& & & S^{n-1} &
\end{tikzcd}
$$

$H_p(U) = 0$, $p \geq 1$ $\qquad$ $H_{p-1}(U) = 0$, $p \geq 2$

For $p \geq 2$, $p \neq n-1, n$, get

$$0 \longrightarrow H_p(X) \xrightarrow{\cong} H_p(Y) \longrightarrow 0 \quad \text{exact}$$

For $p = n-1 \geq 2$, get

$$
\begin{tikzcd}
H_{n-1}(\partial D) \arrow[r, "\varphi_*"] \arrow[dr] & H_{n-1}(X) \arrow[r] & H_{n-1}(Y) \arrow[r] & 0 \quad \text{exact} \\
0 \arrow[r] & \operatorname{im}(\varphi_*) \arrow[u] & &
\end{tikzcd}
$$

short exact

For $p=n$ get

$$\begin{tikzcd}
0 \arrow[r] & H_n X \arrow[r] & H_n Y \arrow[r] \arrow[dr] & H_{n-1}\partial D \arrow[r, "\varphi_*"] & H_{n-1} X \\
& & & \ker \varphi_* \arrow[u] \arrow[dr] & \\
& & & & 0
\end{tikzcd}$$

exact

short exact

If $p=1$,

$$\begin{tikzcd}
H_1(\partial D) \arrow[r, "\varphi_*"] \arrow[dr] & H_1(X) \arrow[r] & H_1(Y) \arrow[r] \arrow[dr] & H_0(\partial D) \arrow[r] & H_0(U) \oplus H_0(X) \\
& \operatorname{im}\varphi_* \arrow[u] & & 0 & \\
0 \arrow[ur] & \pi_1(X,v) \arrow[uu, "\gamma"] \arrow[r, two heads] & \pi_1(Y,v) \arrow[uu, "\gamma"] & &
\end{tikzcd}$$

$H_0(\partial D) \cong \mathbb{Z}$, $\quad H_0(U) \cong$

0 if $n>2$

saw previously

If $p=0$, $H_0 X \cong H_0(Y)$ (gluing doesn't change path components).

Summarizing:

Proposition 13.33 (Homology Effect of Attaching a Cell). *Let X be any topological space, and let Y be obtained from X by attaching a closed cell D of dimension $n \geq 2$ along the attaching map $\varphi \colon \partial D \to X$. Let K and L denote the kernel and image, respectively, of $\varphi_* \colon H_{n-1}(\partial D) \to H_{n-1}(X)$. Then the homology homomorphism $H_p(X) \to H_p(Y)$ induced by inclusion is characterized as follows.*

(a) *If $p < n-1$ or $p > n$, it is an isomorphism.*

(b) *If $p = n-1$, it is a surjection whose kernel is L, so there is a short exact sequence*

$$0 \to L \hookrightarrow H_{n-1}(X) \to H_{n-1}(Y) \to 0.$$

(c) *If $p = n$, it is an injection, and there is a short exact sequence*

$$0 \to H_n(X) \to H_n(Y) \to K \to 0.$$

Thm X a finite n-diml CW complex.

(a) $X_k \hookrightarrow X$ induces $H_p X_k \cong H_p X$ for $p \le k-1$.

(b) $H_p X = 0$ for $p > n$.

(c) For $0 \le p \le n$, $H_p(X)$ is a finitely generated group of rank $\le \#\,p$-cells in X

(d) If X has no cells of dimn $p-1$ or $p+1$, then $H_p(X)$ is free Abelian of rank $= \#\,p$-cells in X.

(e) Suppose X has exactly one cell in dimn n w/ attaching map $\varphi : \partial D \longrightarrow X_{n-1}$. Then

$$H_n X \cong \begin{cases} \mathbb{Z} & \text{if } 0 = \varphi_* : H_{n-1}\partial D \longrightarrow H_{n-1} X_{n-1} \\ 0 & \text{if } \varphi_* \neq 0. \end{cases}$$

A Abelian gp

rank = free rank of A = $\dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q})$ $= r$

$A = \mathbb{Z}^r \oplus$ torsion

(a), (b) : fairly direct from previous thm.

(c) : Immediate if you know that singular and cellular homology agree. O/w

$H_p X \underset{(a)}{\cong} H_p(X_{p+1}) \twoheadleftarrow H_p(X_p)$ by (b) of previous theorem. Thus it suffices to prove rank $H_p X_p \leq \#$ p-cells. Now use (c) of previous thm w/ the fact $K \leq H_{p-1} S^{p-1} \cong \mathbb{Z}$ + rank-nullity.

(d) : Here $H_p X \cong H_p(X_{p+1}) = H_p(X_p)$ b/c $X_p = X_{p+1}$.

Proceed by induction on $m = \#$ p-cells. If $m = 0$, then $H_p(X_p) = 0$ by (c). Suppose $H_p(X_p) \cong \mathbb{Z}^m$ for all X w/ m p-cells. Given X w/ $m+1$ p-cells, let $Z = X \setminus e$ for some p-cell and let

$\varphi : \partial D \longrightarrow X_{p-1} = Z_{p-1}$ be the attaching map for e.

By ind'n hypothesis, $H_p(Z) \cong \mathbb{Z}^m$. By previous prop, have SES

$$0 \longrightarrow H_p Z \longrightarrow H_p X \longrightarrow \underbrace{\ker(\varphi_* : H_{p-1}\partial D \to \underbrace{H_{p-1} X}_{=0})}_{\mathbb{Z}} \longrightarrow 0$$

$\Rightarrow$ rank $H_p X$ = rank $H_p Z$ + 1

$= m+1$.

Cannot have torsion in middle of SES w/ nontorsion on ends. ✓

$$\text{(e)}: \quad 0 \longrightarrow \underbrace{H_{n-1}(X_{n-1})}_{0 \text{ by (d)}} \longrightarrow H_n(X_n) \xrightarrow{\cong} K \longrightarrow 0$$

$K = \ker(\varphi_* : \underbrace{H_{n-1}\partial D}_{\mathbb{Z}} \to H_{n-1} X)$

K: triv if $\varphi_* = 0$, $\cong \mathbb{Z}$ if not

$\square$

E.g. ∘ $\mathbb{RP}^n$ has exactly 1 cell in dimns $0, 1, \ldots, n$ and no higher dim'l cells. $H_n(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$ by (e).

∘ $H_p(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & 0 \leq p \leq 2n \text{ even} \\ 0 & \text{o/w} \end{cases}$ by (d)

single cell in dimns $0, 2, 4, \ldots, 2n$; no others

TPS Compute $H_p(K)$ via : · CW structure
· M-V

Klein bottle

$$H_p K \cong \begin{cases} \mathbb{Z} & p = 0 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & p = 1 \\ 0 & p = 2 \end{cases}$$

Euler characteristic

Euler char of finite CW cpx X w/ n_p cells in dimn p:

$$\chi(X) = \sum_{p \geq 0} (-1)^p n_p .$$

Thm If X is a finite CW cpx, then

$$\chi(X) = \sum_{p \geq 0} (-1)^p \operatorname{rank} H_p(X) .$$

Cor χ is a htpy invariant of finite CW cpxs.

Pf of Thm Assume X conn'd. Proceed by induction on $N = \#$ cells of dimn ≥ 2.

WLOG b/c $H_p Y \cong \bigoplus_{\alpha \in \pi_0 X} H_p Y_\alpha$

$\underline{N=0}$: $\pi_1 X$ is free on $1-\chi(X)$ generators

$$\Longrightarrow H_1 X \cong \mathbb{Z}^{1-\chi(X)}.$$

Further, $H_0 X \cong \mathbb{Z}$, $H_p X = 0$ for $p>1$, so

$$\operatorname{rank} H_0(X) - \operatorname{rank} H_1(X)$$
$$= 1-(1-\chi(X)) = \chi(X)$$

Suppose true for X w/ fewer then N cells for some fixed $N \geq 1$. Consider some X with N cells. For e a cell with max'l dimn (call it n), suffices to show for $Z = X \setminus e$ that

$$\chi(X) = \chi(Z) + (-1)^n.$$

If $\varphi : \partial D \longrightarrow Z$ attaches e, know

$$H_p(X) \cong H_p(Z) \text{ for } p \neq n, n-1$$

and we have exact sequences

$$\operatorname{im}(\varphi_* : H_{n-1}\partial D \to H_{n-1}Z)$$

$$0 \longrightarrow L \longrightarrow H_{n-1}(Z) \longrightarrow H_{n-1}(X) \longrightarrow 0$$

$$0 \longrightarrow H_n(Z) \longrightarrow H_n X \longrightarrow K \longrightarrow 0$$

$$K = \ker(\varphi_* : H_{n-1}\partial D \to H_{n-1}Z)$$

Thus $\operatorname{rank} H_p X = \operatorname{rank} H_p Z$ for $p \neq n, n-1$

$\operatorname{rank} H_{n-1} X = \operatorname{rank} H_{n-1} Z - \operatorname{rank} L$

$\operatorname{rank} H_n X = \operatorname{rank} H_n Z + \operatorname{rank} K$

By SES $0 \to K \to \underset{\mathbb{Z}}{\overset{}{H_{n-1}\partial D}} \to L \to 0$ know $\operatorname{rank} K + \operatorname{rank} L = 1$

$\square$

For X with bounded finite rank homology, define

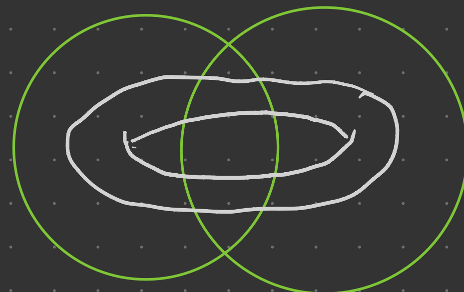
$$\chi(X) := \sum_{p \geq 0} (-1)^p \underbrace{\operatorname{rank} H_p(X)}_{\beta_p(X)}$$

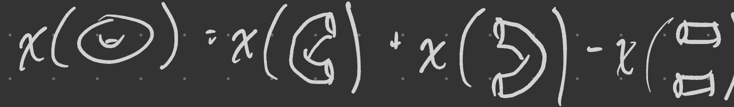
$\beta_p(X)$ — p-th Betti # of X

Euler characteristic facts

- $\chi(S^2) = -2$, $\chi((\mathbb{T}^2)^{\#g}) = 2-2g$, $\chi((\mathbb{RP}^2)^{\#g}) = 2-g$
- $\chi(S^n) = 1 + (-1)^n = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$
- Compact conn'd closed manifold M admits a nowhere vanishing vector field (à la hairy ball) iff $\chi(M) = 0$
- Inclusion-exclusion: $\chi(U \cup V) = \chi(U) + \chi(V) - \chi(U \cap V)$

This is a shadow of Mayer-Vietoris!





$$= 1 + 1 - 2 = 0$$

$$(\text{b/c } \chi(S^1) = 1)$$

- $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$

- $\chi(\circ\!\!-\!\!-\!\!-\!\!\circ) = \chi(\bullet\!\!-\!\!-\!\!-\!\!\bullet) - \chi(\bullet) - \chi(\bullet) = 1 - 1 - 1 = -1$

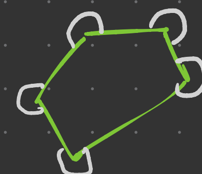
$$\chi(\mathbb{RP}^\infty) = 1 - 1 + 1 - 1 + \cdots \overset{?}{=} \frac{1}{2}$$

$\rightsquigarrow$ "negative" & "fractional" sets (Schanuel, Propp, ...)

- Gauss-Bonnet : M a compact 2-dim'l Riemannian mfld w/ Gaussian curvature K, then

$$\chi(M) = \frac{1}{2\pi}\int_M K \, dA$$

$\therefore$ Integrating a local feature can produce a topological invariant



$H^*(X; \mathbb{Z})$

$C_*(X)$ dualized gives $C^*(X)$

$$0 \to C_n \to \cdots \to C_2 \to C_1 \to C_0 \to 0$$

$$\sum (-1)^k \dim C_k = \sum (-1)^k \dim H_k(C_\bullet)$$