25. I. 23

# Mayer-Vietoris

$U, V \subseteq X$ open, $X = U \cup V$ $\rightsquigarrow$

$$\begin{tikzcd} U \cap V \arrow[r, "i"] \arrow[d, "j"'] \arrow[dr, phantom, "\ulcorner"] & U \arrow[d, "k"] \\ V \arrow[r, "\ell"'] & X \end{tikzcd}$$

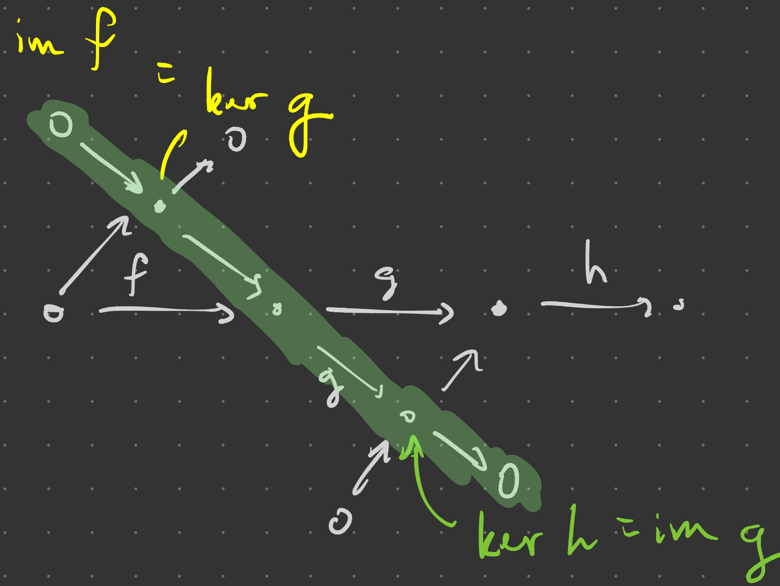
**Thm** (Mayer-Vietoris) For each $p$ $\exists$ homomorphism $\partial_*: H_p(X) \to H_{p-1}(U \cap V)$

s.t. $\cdots \xrightarrow{\partial_*} H_p(U \cap V) \xrightarrow{i_* \oplus j_*} H_p(U) \oplus H_p(V) \xrightarrow{k_* - \ell_*} H_p(X) \xrightarrow{\partial_*} H_{p-1}(U \cap V) \xrightarrow{i_* \oplus j_*} \cdots$ , is exact.

} Mayer-Vietoris sequence

Here a sequence of homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is exact (at $B$) when $\mathrm{im}(f) = \ker(g)$.

Yoga: Long exact sequence + two-thirds of terms $\rightsquigarrow$ all terms.



im f = ker g
ker h = im g
f
g
h
g

Note Short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$

says
- $f$ injective
- $g$ surjective
- $C \cong B/A$

Lemma Let $0 \to C_\bullet \xrightarrow{F} D_\bullet \xrightarrow{G} E_\bullet \to 0$ be a SES of chain cpxs.

Then there are connecting homs $\partial_*: H_p(E_\bullet) \to H_{p-1}(C_\bullet)$ s.t.

$$\cdots \xrightarrow{\partial_*} H_p(C_\bullet) \xrightarrow{F_*} H_p(D_\bullet) \xrightarrow{G_*} H_p(E_\bullet) \xrightarrow{\partial_*} H_{p-1}(C_\bullet) \xrightarrow{F_*} \cdots$$

is exact.

Pf

$$
\begin{tikzcd}
0 \arrow[r] & C_{p+1} \arrow[r, "F"] \arrow[d, "\partial"] & D_{p+1} \arrow[r, "G"] \arrow[d, "\partial"] & E_{p+1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_p \arrow[r, "F"] \arrow[d, "\partial"] & D_p \arrow[r, "G"] \arrow[d, "\partial"] & E_p \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_{p-1} \arrow[r, "F"] \arrow[d, "\partial"] & D_{p-1} \arrow[r, "G"] \arrow[d, "\partial"] & E_{p-1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_{p-2} \arrow[r, "F"] & D_{p-2} \arrow[r, "G"] & E_{p-2} \arrow[r] & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & d \arrow[r, mapsto, "G"] \arrow[d, mapsto] & e \arrow[d, mapsto] \\
c \arrow[r, mapsto, "F"] \arrow[d, mapsto] & \partial d \arrow[r, mapsto, "G"] \arrow[d, mapsto] & 0 \\
\partial c \arrow[r, mapsto, "F"] & 0 &
\end{tikzcd}
$$

$\partial c = 0$ since $F$ injective

$0$ b/c $\partial\partial = 0$

Squares commute, rows exact

$$[e] \in H_p(E_\bullet) \xrightarrow[\partial_*]{} [c] \in H_{p-1}(C_\bullet) = Z_{p-1} / B_{p-1}$$

Well-defined hom: check / read p. 357

Exactness at $H_p(C_\bullet)$: Suppose $[c] = \partial_*[e]$, $e \in E_{p+1}$.

Then $\exists\, d \in D_{p+1}$ s.t. $Fc = \partial d \Rightarrow F_*[c] = [Fc] = [\partial d] = 0$

Thus $\operatorname{im} \partial_* \subseteq \ker F_*$. Now if $F_*[c] = [Fc] = 0$, then $Fc = \partial d$ for some $d \in D_{p+1}$, whence $\partial G d = G \partial d = G F c = 0$.

Thus $[Gd] \in H_{p+1}(E_\bullet)$ and $\partial_*[Gd] = [c]$ so $\ker F_* \subseteq \operatorname{im} \partial_*$.

$$
\begin{tikzcd}
0 \arrow[r] & C_{p+1} \arrow[r, "F"] \arrow[d, "\partial"'] & D_{p+1} \arrow[r, "G"] \arrow[d, "\partial"'] & E_{p+1} \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & C_p \arrow[r, "F"] & D_p \arrow[r, "G"] & E_p \arrow[r] & 0
\end{tikzcd}
$$

$d \longmapsto e$

$c \longmapsfrom Fc = \partial d \mapsto GFc = 0$

Exactness at other spots: check / read p. 358 $\square$

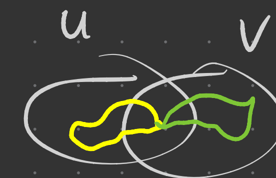
## (Partial) Pf of Mayer-Vietoris

$$\begin{tikzcd} U\cap V \arrow[r, "i"] \arrow[d, "j"'] \arrow[dr, phantom, "\ulcorner"] & U \arrow[d, "k"] \\ V \arrow[r, "l"'] & X \end{tikzcd}$$

We would like to apply the lemma to chain cpxs

$$C_\bullet(U\cap V), \quad C_\bullet(U)\oplus C_\bullet(V), \quad C_\bullet(X).$$

We do have an exact sequence

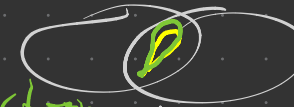



$$0 \longrightarrow C_p(U\cap V) \xrightarrow{i_\# \oplus j_\#} C_p(U)\oplus C_p(V) \xrightarrow{k_\# - l_\#} C_p(X) \;:$$

- The meaning of $(i_\# \oplus j_\#)(c) = 0$ is that $c$ is $0$ on $U$ and $0$ on $V$ hence $0$ on $U\cap V$, so $c=0$ and $i_\# \oplus j_\#$ is injective.
- If $(k_\# - l_\#)(c,d) = k_\# c - l_\# d = 0$ then $k_\# c = l_\# d$ i.e. $c, d$ agree on $X$. Their restriction to $U\cap V$ then maps to

$(c,d)$ under $i_\# \oplus j_\#$.

Don't escape $U \cap V$ by cartoon



- But $k_\# - l_\#$ is not surjective: a singular $p$-simplex with image not all in $U$ or all in $V$ is not in $\operatorname{im}(k_\# - l_\#)$.

Subterfuge For $\mathcal{U} = \{U, V\}$ open cover, let $C_p^{\mathcal{U}}(X)$ = subgp of $C_p(X)$ gen'd by singular simplices w/ image all in $U$ or all in $V$.

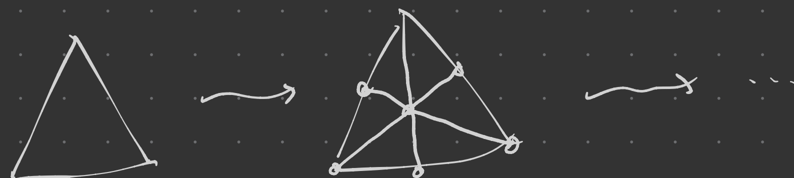
Get SES $0 \longrightarrow C_p(U \cap V) \xrightarrow{i_\# \oplus j_\#} C_p(U) \oplus C_p(V) \xrightarrow{k_\# - l_\#} C_p^{\mathcal{U}}(X) \longrightarrow 0$

and, by lemma, induced LES in homology of desired shape but with $H_p(C_\bullet^{\mathcal{U}}(X))$ in place of $H_p(X)$.

WTS $C_\bullet^{\mathcal{U}}(X) \hookrightarrow C_\bullet(X)$ induces a homology isomorphism.

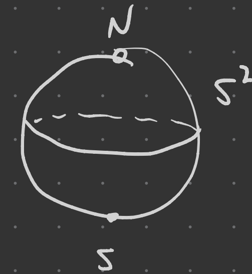
For $\mathcal{U}$ an arbitrary open cover of $X$, call a singular chain $c$ $\mathcal{U}$-small if all its singular simplices have images in one of the sets of $\mathcal{U}$. Define $C_p^{\mathcal{U}}(X) \subseteq C_p(X)$ as all $\mathcal{U}$-small chains

Show that $C_\bullet^{\mathcal{U}}(X) \hookrightarrow C_\bullet(X)$ induces homology iso via subdivision: break up $\Delta_p$ into smaller & smaller pieces until each piece is $\mathcal{U}$-small.



For the diligent: pp. 360-364

Let's use M-V:




$$S^n = U \cup V = \underbrace{(S^n \setminus N)}_{\cong \mathbb{R}^n \simeq *} \cup \underbrace{(S^n \setminus S)}_{\cong \mathbb{R}^n \simeq *}$$

$$U \cap V = S^n \setminus \{N, S\} \cong \mathbb{R}^n \setminus 0 \simeq S^{n-1}$$

$p > 1$

$$\cdots \to \underbrace{\cancel{H_p(U)} \oplus \cancel{H_p(V)}}_{0} \longrightarrow H_p(S^n) \xrightarrow{\cong} H_{p-1}(U \cap V) \longrightarrow \underbrace{\cancel{H_{p-1}(U)} \oplus \cancel{H_{p-1}(V)}}_{0} \to \cdots$$

$$H_{p-1}(U \cap V) \cong H_{p-1}(S^{n-1})$$

$$0 \to A \xrightarrow{\cong} B \to 0$$

Already know $H_0 S^n \cong \mathbb{Z}$ for $n \geq 1$ (path conn'd)

and $H_1 S^n \cong \begin{cases} \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{if } n > 1 \text{ (simply conn'd.} \end{cases}$

Also for $p > 1$, $H_p(S^1) \cong H_{p-1}(S^0) = 0$ since $S^0$ discrete.

Now know $H_* S^1 = \begin{array}{cccccc} \mathbb{Z} & \mathbb{Z} & 0 & 0 & 0 & \cdots \\ 0 & 1 & 2 & 3 & 4 & \end{array}$

For $S^2$ have $\mathbb{Z}$ $0$ $?$ $?$ $?$ $\cdots$

but $H_p S^2 \cong H_{p-1} S^1$ for $p>1$ so the pattern is

$\mathbb{Z}$ $0$ $\mathbb{Z}$ $0$ $0$ $\cdots$.

Let's fill in the table for $H_p(S^n)$:

| $n \backslash p$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | $\mathbb{Z}^2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | $\mathbb{Z}$ | $\mathbb{Z}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 | 0 | 0 |
| 3 | $\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}$ | 0 | 0 | 0 | 0 |
| 4 | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |

**Thm** For $n \geq 1$, $H_p(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } p=0 \\ 0 & 0<p<n \\ \mathbb{Z} & p=n \\ 0 & p>n \end{cases}$

**TPS** Use M-V to compute $H_p(S^1 \vee S^1)$.

What about $H_p(X \vee Y)$ ?

**Degree Theory for spheres**

$f: S^n \longrightarrow S^n$ continuous induces $f_*: \underset{\mathbb{Z}}{\overset{\cong}{H_n(S^n)}} \longrightarrow \underset{\mathbb{Z}}{\overset{\cong}{H_n(S^n)}}$

$(n \geq 1)$

**Fact** $\text{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$.

$\underbrace{(n \mapsto kn)}_{= m_k} \longleftarrow k$

**Defn** The **degree** of a cts map $f: S^n \longrightarrow S^n$ $(n \geq 1)$ is the unique integer $\deg(f)$ s.t. $f_* = m_{\deg(f)}$.

By functoriality & htpy invariance,

- $\deg(f \circ g) = \deg(f) \cdot \deg(g)$
- $f \simeq g \Rightarrow \deg(f) = \deg(g)$

**Computations**

- $\deg(id: S^n \longrightarrow S^n) = 1$
- $\deg(\text{const}) = 0$
- $\deg(\text{reflection}: S^n \longrightarrow S^n) = -1$ (ref'n through plane through 0)
- $\deg(\text{antipodal}: S^n \longrightarrow S^n) = (-1)^{n+1}$

- For reflection, use $R_1 : (x_1, \dots, x_{n+1}) \longmapsto (-x_1, x_2, \dots, x_{n+1})$.

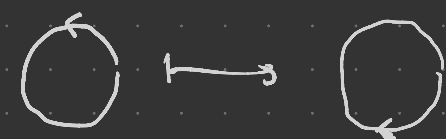
Homotopic to any other reflection by rotating plane reflected through.

Now

$$\begin{tikzcd} H_n(S^n) \arrow[r, "\partial_*"', "\cong"] \arrow[d, "R_{1*}"'] & H_{n-1}(S^{n-1}) \arrow[d, "R_{1*}"] \\ H_n(S^n) \arrow[r, "\partial_*"', "\cong"] & H_{n-1}(S^{n-1}) \end{tikzcd}$$

naturality of $\partial_*$ ; see text p. 367

So, by induction, suffices to check $\deg(R_{1*}) = -1$ on $S^1$:



- Antipodal $= R_1 \circ R_2 \circ \cdots \circ R_{n+1}$ for

$$R_i : (x_1, \dots, x_{n+1}) \longmapsto (x_1, \dots, -x_i, \dots, x_{n+1}).$$

**Prop** The antipodal map $\alpha : S^n \longrightarrow S^n$ is htpic to id iff $n$ is odd.

**Pf** If $n = 2k-1$ is odd, then $H : \mathrm{id} \simeq \alpha$ is given by

$$H(x,t) = \big((\cos \pi t)x_1 + (\sin \pi t)x_2,\ (\cos \pi t)x_2 - (\sin \pi t)x_1,$$
$$\ldots,\ (\cos \pi t)x_{2k-1} + (\sin \pi t)x_{2k},\ (\cos \pi t)x_{2k} - (\sin \pi t)x_{2k-1}\big).$$

If $n=0$, swapping 2 pts is not htpic to id.

If $n>0$ is even, $\deg \alpha = -1$ while $\deg \mathrm{id} = 1$. $\square$

A **vector field** on $S^n$ is a cts map $V : S^n \longrightarrow \mathbb{R}^{n+1}$ s.t.

$\forall x \in S^n$, $(Vx) \cdot x = 0$ (so $Vx$ is tangent to $S^n$ at $x$)


**Thm** There exists a $\underbrace{\text{nowhere vanishing}}_{Vx \neq 0\ \forall x}$ vector field on $S^n$

iff $n$ is odd.

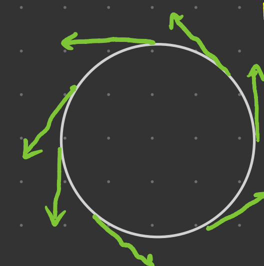
└ "hairy ball" or "hedgehog" theorem

Pf Suppose $V$ is a nowhere vanishing vector field on $S^n$. WLOG ($V/|V|$) may assume $|Vx| = 1$ $\forall x$. Define

$$\begin{aligned} H : S^n \times I &\longrightarrow S^n \\ (x, t) &\longmapsto (\cos \pi t)x + (\sin \pi t)Vx. \end{aligned}$$

TPS Check $H(x,t) \in S^n$

$n=1$



Have $H(x,0) = x$, $H(x,1) = -x$ so $H : \mathrm{id} \simeq \alpha$.

By the prop'n, $n$ must be odd.

If $n = 2k-1$, $V(x_1, \dots, x_{2k}) = (x_2, -x_1, x_4, -x_3, \dots, x_{2k}, -x_{2k-1})$ works. $\square$