22.I.23

Homotopy Invariance of Homology

Thm If $f_0, f_1 : X \longrightarrow Y$ are homotopic, then $\forall p \geq 0$, the maps $f_{0*}, f_{1*} : H_p(X) \longrightarrow H_p(Y)$ are equal.

I.e. $H_p : \mathrm{Top} \longrightarrow \mathrm{Ab}$ factors through Hot.

Cor If $f : X \xrightarrow{\simeq} Y$ is a htpy equivalence, then $f_* : H_p(X) \to H_p(Y)$ is an isomorphism. $\square$

Pf of Thm Claim It suffices to show $id \times 0, id \times 1 : X \to X \times I$ induce the same maps on H_p.

Pf Claim Suppose $H : f_0 \simeq f_1$. Then

$$\begin{tikzcd}
X \times 0 \arrow[d] \arrow[dr, bend left, "f_0"] & \\
X \times I \arrow[r, "H"] & Y \\
X \times 1 \arrow[u] \arrow[ur, bend right, "f_1"'] &
\end{tikzcd}$$

$$(f_0)_* = (H\circ(id\times 0))_* = H_*\circ(id\times 0)_*$$
$$(f_1)_* = (H\circ(id\times 1))_* = H_*\circ(id\times 1)_*$$

equal if $(id\times 0)_* = (id\times 1)_*$ ✓

We prove that $(id\times 0)_* = (id\times 1)_*$ by proving that

$$(id\times 0)_\#,\ (id\times 1)_\# : C_*(X) \longrightarrow C_*(X\times I)$$

are chain homotopic.

Homological Algebra Interlude

Chain complexes
$$C_\bullet = \cdots \xrightarrow{\partial} C_{p+1} \xrightarrow{\partial} C_p \xrightarrow{\partial} C_{p-1} \xrightarrow{\partial} \cdots$$
$$D_\bullet = \cdots \xrightarrow[\partial]{} D_{p+1} \xrightarrow[\partial]{} D_p \xrightarrow[\partial]{} D_{p-1} \xrightarrow[\partial]{} \cdots$$

Chain maps $F, G : C_\bullet \longrightarrow D_\bullet$ are chain homotopic when
$(F\partial = \partial F,\ G\partial = \partial G)$

$\exists$ chain homotopy $\{h : C_p \to D_{p+1}\}_p$, a collection of homomorphisms satisfying $h\circ\partial + \partial\circ h = G - F$:

$$\begin{tikzcd} \cdots & C_p \arrow[r, "\partial"] \arrow[dl, "h"'] \arrow[d, "G-F"] & C_{p-1} \arrow[dl, "h"] \\ D_{p+1} \arrow[r, "\partial"'] & D_p & \end{tikzcd}$$

Aside Possible to interpret h as $h : C_\bullet \otimes I_\bullet \longrightarrow D_\bullet$

for $I_\bullet = (\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{(id,\,-id)} \mathbb{Z}\oplus\mathbb{Z})$.

$\ker(\partial : C_p \to C_{p-1})$
$\|$

Suppose $h : F \simeq G$ is a chain homotopy. Then for $c \in Z_p(C)$,

$$Gc - Fc = h\cancelto{0}{\partial c} + \partial h c = \partial h c \implies G_*[c] = F_*[c]$$

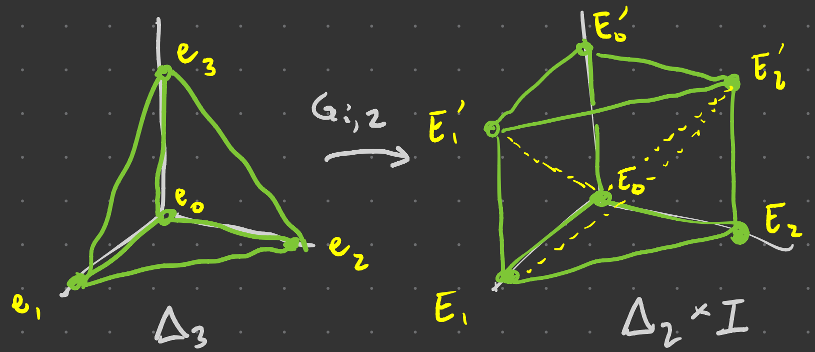
(for $[c]$ = class of c in $H_p(C) = Z_p(C)/B_p(C)$).

Thm If $F, G : C_\bullet \to D_\bullet$ are chain homotopic chain maps, then $F_* = G_* : H_p(C_\bullet) \to H_p(D_\bullet)$ $\forall p$. $\square$

Back to homotopy invariance, it now suffices to construct a chain homotopy $h : C_p(X) \to C_{p+1}(X \times I)$ satisfying

$$h\partial + \partial h = (id \times 1)_\# - (id \times 0)_\# .$$

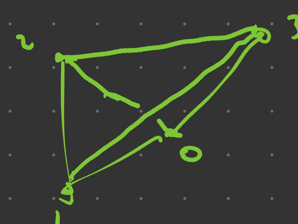
Here: $p = 2$. Reading: gen'l case (pp. 348-350).



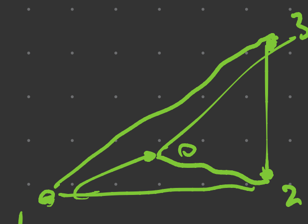
$G_{0,2} : \Delta_3 \longrightarrow \Delta_2 \times I$ affine with

$$\begin{aligned} e_0 &\longmapsto E_0 \\ e_1 &\longmapsto E_0' \\ e_2 &\longmapsto E_1' \\ e_3 &\longmapsto E_2' \end{aligned}$$

$G_{1,2}$:
$$\begin{aligned} e_0 &\longmapsto E_0 \\ e_1 &\longmapsto E_1 \\ e_2 &\longmapsto E_1' \\ e_3 &\longmapsto E_2' \end{aligned}$$



$G_{2,2}$:
$$\begin{aligned} e_0 &\longmapsto E_0 \\ e_1 &\longmapsto E_1 \\ e_2 &\longmapsto E_2 \\ e_3 &\longmapsto E_2' \end{aligned}$$



Then define, for $\sigma : \Delta^p \longrightarrow X$,

$$h(\sigma) := \sum_{i=0}^{p} (-1)^i (\sigma \times Id) \circ G_{i,p}$$

$$= (\sigma \times Id) \circ G_{0,2} - (\sigma \times Id) \circ G_{1,2} + (\sigma \times Id) \circ G_{2,2}$$

$(p=2)$

and extend linearly to get $h : C_p(X) \longrightarrow C_{p+1}(X \times I)$

Lemma $(F_{j,p} \times Id) \circ G_{i,p+1} = \begin{cases} G_{i+1,p} \circ F_{j,p+1} & \text{if } i \geq j \\ G_{i,p} \circ F_{j+1,p+1} & \text{if } i < j \end{cases}$

Pf TPS $\square$

Thus $h(\partial\sigma) = h\sum_{j=0}^{p}(-1)^j \sigma\circ F_{j,p}$

$$= \sum_{i=0}^{p-1}\sum_{j=0}^{p}(-1)^{i+j}\left(\left(\sigma\circ F_{j,p}\right)\times Id\right)\circ G_{i,p-1}$$

$$= \sum_{i=0}^{p-1}\sum_{j=0}^{p}(-1)^{i+j}\left(\sigma\times Id\right)\circ\left(F_{j,p}\times Id\right]\circ G_{i,p-1}$$

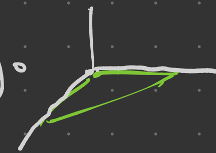
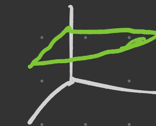
$$= \sum_{0\le j\le i\le p-1}(-1)^{i+j}\left(\sigma\times Id\right)\circ G_{i+1,p}\circ F_{j,p+1}$$

$$+ \sum_{0\le i<j\le p}(-1)^{i+j}\left(\sigma\times Id\right)\circ G_{i,p}\circ F_{j+1,p+2}$$

Similar manipulations w/ $\partial h \sigma$ yield spectacular cancellation

(p. 350)

and ultimately

$$h(\partial\sigma) + \partial h(\sigma) = -(\sigma \times Id) \circ$$  $$+ (\sigma \times Id) \circ$$ 

$$= -(id \times 0)_{\#}\,\sigma + (id \times 1)_{\#}\,\sigma$$

as desired. $\square$ /details.

$$\begin{tikzcd} Top \arrow[r, "H_p"] \arrow[d] & Ab \\ Hot \arrow[ur, dashed] & \end{tikzcd}$$!

Homology & π_1

Loop $f: I \longrightarrow X$ = singular 1-simplex $f: \Delta^1 \longrightarrow X$

w/ $\partial f = \underbrace{f(1) - f(0)}_{=0} \in C_0(X)$

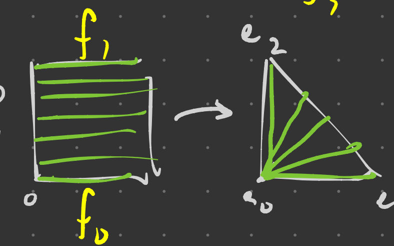
formal

$\Longrightarrow$ Loops are 1-cycles

Lemma Suppose $f_0 \sim f_1$ are path htpic paths in X. Then $f_0 - f_1 \in B_1(X)$.

Pf Let $H : f_0 \sim f_1$, $b : I^2 \longrightarrow \Delta_2$ quotient map

$$(x,y) \longmapsto (x - xy, xy)$$



Have

$$\begin{tikzcd} I^2 \arrow[r, "H"] \arrow[d, "b"'] & X \\ \Delta_2 \arrow[ur, dashed, "\sigma"'] & \end{tikzcd}$$

with $\partial\sigma = c_p - f_1 + f_0$ for $p = f_0(1) = f_1(1)$

We have $\sigma' : \Delta^2 \longrightarrow X$ a sing 2-spx w/ $\partial\sigma' = c_p - c_p + c_p$
$(x,y) \longmapsto p$ $\qquad = c_p$.

Thus $f_0 - f_1 = \partial(\sigma - \sigma') \in B_1(X)$ $\square$

Defn The Hurewicz homomorphism

$$\gamma_X = \gamma : \pi_1(X,p) \longrightarrow H_1(X)$$
$$\underbrace{[f]_\pi}_{\text{path htpy class of } f} \longmapsto \underbrace{[f]_H}_{\text{homology class of the 1-cycle } f}$$

TPS The following "naturality diagram" commutes:

for $F : X \to Y$ cts,

$$\begin{tikzcd}
\pi_1(X,p) \arrow[r, "F_*"] \arrow[d, "\gamma_X"'] & \pi_1(Y, F(p)) \arrow[d, "\gamma_Y"] \\
H_1(X) \arrow[r, "F_*"'] & H_1(Y)
\end{tikzcd}$$

$$\sum n_i \sigma_i \longmapsto \sum n_i (F \circ \sigma_i)$$

$$\begin{tikzcd}
\Delta^1 \arrow[r, "\sigma_i"] \arrow[rr, bend right] & X \arrow[r, "F"] & Y
\end{tikzcd}$$

$$\gamma : \pi_1(\;, p) \Rightarrow H_1$$

$$\begin{tikzcd}
{[f]_\pi} \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & {[F \circ f]_\pi} \arrow[ddd, mapsto] \\
& \pi_1(X, p) \arrow[r, "F_*"] \arrow[d, "\gamma_X"'] & \pi_1(Y, F(p)) \arrow[d, "\gamma_Y"] & \\
& H_1(X) \arrow[r, "F_*"'] & H_1(Y) & \\
{[f]_H} \arrow[rrr, mapsto] & & & {[F \circ f]_H}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{D}
\end{tikzcd}$$

$$F, G : \mathcal{C} \longrightarrow \mathcal{D}$$

$f \in \mathcal{C}(x, y) \rightsquigarrow$

$$\begin{tikzcd}
Fx \arrow[r, "Ff"] \arrow[d, "\alpha_x"'] \arrow[dr, phantom, "\star"] & Fy \arrow[d, "\alpha_y"] \\
Gx \arrow[r, "Gf"'] & Gy
\end{tikzcd}$$

$\alpha : F \Rightarrow G$ is a natural transformation when $\forall x \in \mathrm{Ob}\, \mathcal{C}$, have $\alpha_x : Fx \to Gx$ st. $\star$ commutes.

Thm Let X be a path conn'd space, $p \in X$. Then $\gamma : \pi_1(X,p) \longrightarrow H_1(X)$ is surjective w/ kernel $[\pi_1(X,p), \pi_1(X,p)]$. Thus γ exhibits $H_1(X)$ as the Abelianization ~~of~~ of $\pi_1(X,p)$.

Pf Surj homomorphism: pp. 353–354.

(Key for surj: choose path $\alpha(x) : p \rightsquigarrow x$, define $\tilde{\sigma} = \alpha(\sigma(0)) \cdot \sigma \cdot \overline{\alpha(\sigma(1))}$. Given $c = \sum_{i=1}^{m} n_i \sigma_i$, use $f = (\tilde{\sigma}_1)^{n_1} \cdots (\tilde{\sigma}_m)^{n_m}$ and show $[f]_H = [c]_H - [\alpha(\partial c)]_H$)

Set $\Pi = \pi_1(X,p)^{ab}$ and write $\pi_1(X,p) \longrightarrow \Pi$ for univ map.

$$[f]_\pi \longmapsto [f]_\Pi$$

For $\sigma : \Delta^1 \longrightarrow X$ sing 1-spx, let $\beta(\sigma) = [\tilde{\sigma}]_\Pi \in \Pi$.

Since Π is Abelian, get $\beta : C_1(X) \longrightarrow \Pi$ extending.

(a) Show $B_1(X) \subseteq \ker \beta$ p.354

(b) For $[f]_\pi \in \ker \gamma$, have $[f]_H = 0 \implies f \in B_1(X)$.

Thus $\beta(f) = [\tilde{f}]_\pi \underset{f \text{ is a loop}}{=} [f]_\pi = 1$, i.e. $[f]_\pi \in$ commutator subgp. $\square$

Cor $H_1 S^2 = 0$

$$H_1((\mathbb{T}^2)^{\# n}) = \mathbb{Z}^{2n}$$

$$H_1((\mathbb{RP}^2)^{\# n}) = \mathbb{Z}/2 \oplus \mathbb{Z}^{n-1}$$