20. I. 23

Singular Homology

π_1 = "loops / loops bounding disks"

H_1 = "loops / loops bounding more general "surfaces""

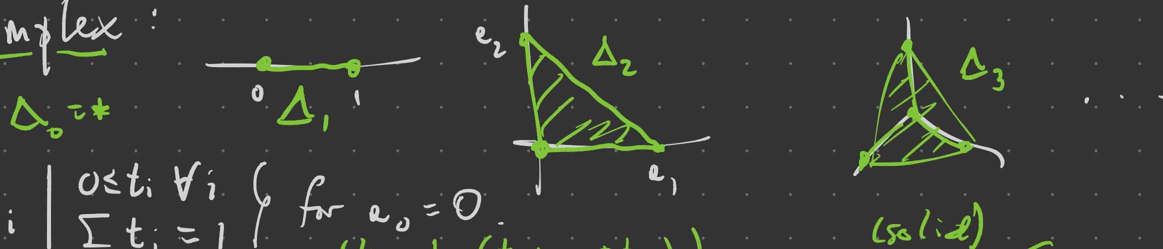
$\underbrace{H_n(X)}_{}$, $n \in \mathbb{N}$, is a sequence of Abelian groups that is

n-th (singular) homology of X

- functorial
- homotopy invariant
- computable
- useful.

For $p \in \mathbb{N}$, $\Delta_p = \left\{ (t_1, \dots, t_p) \in \mathbb{R}^p \;\middle|\; \begin{array}{l} 0 \le t_i \ \forall i \\ \sum t_i \le 1 \end{array} \right\} \subseteq \mathbb{R}^p$ is the

standard p-simplex:



Note $\Delta_p = \left\{ \sum_{i=0}^{p} t_i e_i \;\middle|\; \begin{array}{l} 0 \le t_i \ \forall i \\ \sum t_i = 1 \end{array} \right\}$ for $e_0 = 0$.

$(t_0 = 1 - (t_1 + \cdots + t_p))$

A singular p-simplex in a space X is a cts map

$$\sigma : \Delta_p \longrightarrow X$$

Think of σ as a (potentially degenerate) simplex in X. "Singular" reminds us that σ need not be an embedding

The singular chain group in dimension p is

$$\mathbb{Z} \cdot \{ \sigma \mid \sigma : \Delta_p \to X \text{ cts} \} =: C_p(X)$$

i.e. the free Abelian group on singular p-simplices of X. A generic elt looks like $\sum_{i \in I} n_i \sigma_i$, I finite, $n_i \in \mathbb{Z}$.

Goal Isolate p-chains that "close up on themselves" modulo closed chains that are "boundaries" of chains one dimension higher.

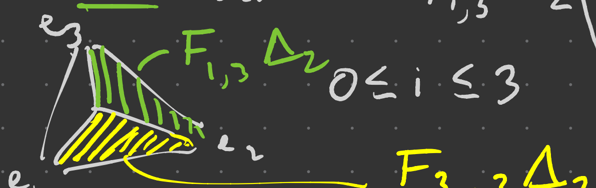
Define i-th face map in dimension p.

$$F_{i,p} : \Delta_{p-1} \longrightarrow \Delta_p \qquad \text{for } 0 \leq i \leq p.$$

$$\sum_{0\leq j\leq p-1} t_j e_j \longmapsto \sum_{0 \leq j \leq i-1} t_j e_j + \sum_{i+1 \leq j \leq p} t_{j-1} e_j$$

$$t_0 e_0 + t_1 e_1 \xrightarrow[F_{0,2}]{} t_0 e_1 + t_1 e_2$$

TPS Describe $F_{i,3}\Delta_2$ $\quad 0 \leq i \leq 3$



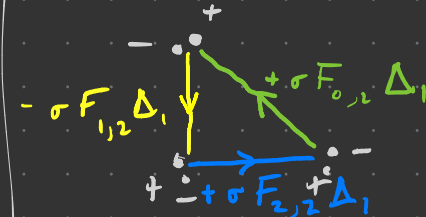
$$t_0e_0 + t_1e_1 + t_2e_2 \longmapsto t_0e_0 + t_1e_2 + t_2e_3$$

For a singular p-simplex $\sigma : \Delta_p \to X$, define $\partial\sigma \in C_{p-1}(X)$, the **boundary** of σ, by

$$\partial\sigma := \sum_{i=0}^{p} \underbrace{(-1)^i}_{\text{"orients" the boundary}} \underbrace{\sigma \circ F_{i,p}}_{\sigma|_{i\text{-th face of } \Delta_p}}$$

$$\boxed{\partial^2 = 0}$$

E.g.: $\sigma : \Delta_2 \hookrightarrow \mathbb{R}^2$



TPS $\partial(\partial\sigma)$ in this case? $= 0$

Extend ∂ linearly to get $\underbrace{\partial}_{=\partial_p} : C_p(X) \longrightarrow C_{p-1}(X)$

$$\sum n_k \sigma_k \longmapsto \sum n_k \partial\sigma_k$$

Set $\partial_0 := 0$ and $C_{-1}(X) := 0$.

Cycles $Z_p(X) := \ker(\partial_p)$

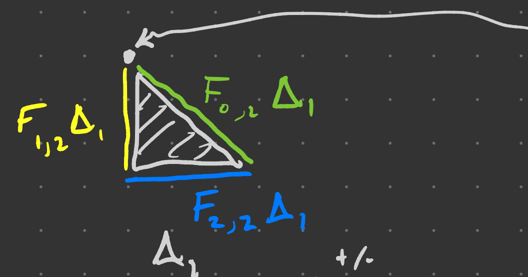
Boundaries $B_p(X) := \mathrm{im}(\partial_{p+1})$

both subgroups of $C_p(X)$.

Lemma For all $c \in C_p(X)$, $\partial^2 c = \partial(\partial c) = 0$.

I.e. $B_p(X) \leq Z_p(X)$.

Pf Intensive bookkeeping, first observing $F_{i,p} \circ F_{j,p-1} = F_{j,p} \circ F_{i-1,p-1}$ for $i>j$: "i-th face of the j-th face $=$ j-th face of the $(i-1)$-th face"



$F_{1,2} \circ F_{0,1} = F_{0,2} \circ F_{0,1}$

etc.

$$\partial(\partial(\sigma : \Delta_2 \hookrightarrow \mathbb{R}^2)) = \sum \begin{pmatrix} \overset{+/-}{\bullet\bullet} & \\ & \\ \underset{+/-}{\bullet\bullet} & \underset{+/-}{\bullet\bullet} \end{pmatrix} = 0.$$

p. 342 $\square$

The p-th singular homology group of X is defined to be

$$H_p X := Z_p(X) \big/ B_p(X)$$

$$= \frac{\ker\left(\partial_p : C_p(X) \to C_{p-1}(X)\right)}{\operatorname{im}\left(\partial_{p+1} : C_{p+1}(X) \to C_p(X)\right)}.$$

Topological invariance follows from functoriality:

$$f : X \to Y \text{ cts} \rightsquigarrow f_\# : C_p(X) \to C_p(Y)$$

$$\sigma \longmapsto f \circ \sigma \quad \text{extended linearly}$$

Have $f_\#(\partial\sigma) = f_\#\left(\sum_{i=0}^{p} (-1)^i \sigma \circ F_{i,p}\right) = \sum_{i=0}^{p} (-1)^i f \circ \sigma \circ F_{i,p} = \partial(f_\# \sigma)$

so $f_\# \circ \partial = \partial \circ f_\#$.

Some language: $\cdots \longrightarrow C_{p+1}(X) \xrightarrow{\partial} C_p(X) \xrightarrow{\partial} C_{p-1}(X) \to \cdots$

is a chain complex ($\partial^2 = 0$) and $f_\#$ is a chain homomorphism (map)

(squares commute)

$$\begin{tikzcd}
\cdots \arrow[r] & C_{p+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"'] & C_p(X) \arrow[r, "\partial"] \arrow[d, "f_\#"'] & C_{p-1}(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r] & C_{p+1}(Y) \arrow[r, "\partial"'] & C_p(Y) \arrow[r, "\partial"'] & C_{p-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
c \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\
f_\# c \arrow[r, mapsto] & \partial f_\# c = 0
\end{tikzcd}$$

TPS It follows that $f_\# Z_p(X) \leq Z_p(Y)$

and $f_\# B_p(X) \leq B_p(Y)$

so $f_\#$ passes to quotients inducing

$$\begin{tikzcd}
Z_p(X) \arrow[r, "f_\#"] \arrow[d] & Z_p(Y) \arrow[d] \\
H_p(X) \arrow[r, dashed] & H_p(Y)
\end{tikzcd}$$

$f_* : H_p(X) \longrightarrow H_p(Y)$, the homomorphism induced by f.

Moreover, $\mathrm{Top} \xrightarrow{H_p} \mathrm{Ab}$ is a functor:

$$\begin{tikzcd} X \arrow[r, mapsto] & H_p(X) \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[d, "f"'] & & H_p(X) \arrow[d, "f_* = H_p(f)"] \\ Y & \arrow[l, phantom] \arrow[r, mapsto] & H_p(Y) \end{tikzcd}$$

- $(\mathrm{id}_X)_* = \mathrm{id}_{H_p(X)}$
- $(g \circ f)_* = g_* \circ f_*$ for $X \xrightarrow{f} Y \xrightarrow{g} Z$ cts, composable.

Thus
- H_p takes homeomorphisms to isomorphisms, and
- retracts $A \subseteq X$ to injections.

  $\left(\begin{matrix} r \\ r\iota = \mathrm{id} \end{matrix} \right)$

Computations (easy mode)

Prop X a space w/ path components $\{X_\alpha \mid \alpha \in \pi_0 X\}$, and let $\iota_\alpha : X_\alpha \hookrightarrow X$. Then $\bigoplus_{\alpha\in\pi_0 X} (\iota_\alpha)_\# : \bigoplus_{\alpha\in\pi_0 X} C_p(X_\alpha) \to C_p(X)$

and $\bigoplus (i_\alpha)_* : \bigoplus H_p(X_\alpha) \longrightarrow H_p(X)$ are isomorphisms.

Pf
$$\begin{tikzcd} \Delta_p \arrow[rr, "\sigma"] \arrow[dr, dashed] & & X \\ & X_\alpha \arrow[ur, "\iota_\alpha"'] & \end{tikzcd}$$
for some α $\square$

Prop For any space X, $H_0(X) \cong \mathbb{Z}^{\pi_0 X}$

Pf By previous prop, suffices to show $H_0(X) \cong \mathbb{Z}$ for X path conn'd.

$$C_0(X) = \{ \sum^{\text{finite}} n_i x_i \mid n_i \in \mathbb{Z}, x_i \in X \} \xrightarrow{\partial} 0$$

so $Z_0(X) = C_0(X)$. Now for X path conn'd, define

$$\varepsilon : C_0(X) \longrightarrow \mathbb{Z} \quad \text{surj hom.}$$
$$\sum n_i x_i \longmapsto \sum n_i$$

WTS $\ker \varepsilon = B_0(X)$ whence $H_0(X) \cong \mathbb{Z}$ by first iso thm.

For $\sigma \in C_1(X)$, $\partial \sigma = \sigma(1) \overset{\text{formal}}{-} \sigma(0) \overset{\varepsilon}{\longmapsto} 1 - 1 = 0$.

Thus $B_0(X) \subseteq \ker \varepsilon$.

Now take $c = \sum n_i x_i \in C_0(X)$. By fixing $x_0 \in X$ and choosing path $\alpha(x) : x_0 \rightsquigarrow x$ for each x, get

$$\partial\Big(\sum_i n_i \alpha(x_i) \Big) = \sum n_i x_i - \sum n_i x_0 = c - \varepsilon(c) x_0 .$$

If $\varepsilon(c) = 0$, then $c = \partial\left(\sum n_i\, \alpha(x_i) \right) \in B_0(X)$

so $\ker \varepsilon \subseteq B_0(X)$. This finishes the proof. $\square$

Finally, let's compute the homology of a point $*$.

Have $C_p(*) = \mathbb{Z}\{ \sigma_p : \Delta_p \to * \} \cong \mathbb{Z}$ $\forall p$, and

$$\partial \sigma_p = \sum_{i=0}^{p} (-1)^i \sigma_p \circ F_{i,p} = \sum_{i=0}^{p} (-1)^i \sigma_{p-1} = \begin{cases} 0 & p \text{ odd or } 0 \\ \sigma_{p-1} & p > 0 \text{ even} \end{cases}$$

The singular chain complex is thus

$$\cdots \xrightarrow{\cong} C_3(*) \xrightarrow{0} C_2(*) \xrightarrow{\cong} C_1(*) \xrightarrow{0} C_0(*) \to 0$$

$$\Rightarrow H_p(*) \cong \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$$

TPS $H_p(\text{discrete space})$?