Classification Theorem X a space with a universal covering space (e.g. conn'd loc simply conn'd), $x_0 \in X$ any base point. There is a bijection $\{q : E \to X \mid q \text{ covering}\}/_{\text{covering iso}} \cong \mathrm{Sub}(\pi_1(X, x_0))/_{\text{conjugacy}}$

$q \longmapsto$ conj class of $q_* \pi_1(E, e)$ for $q(e) = x_0$.

$$\boxed{\text{Here } \mathrm{Sub}(G) := \{H \mid H \leq G\} \text{ is the subgroup lattice of } G}$$

$$\begin{tikzcd} E_H \arrow[r, dotted, no head] \arrow[dr] & E_K \arrow[d] \\ & X \end{tikzcd}$$

$H \leq K \leq G$

Note For a version without conjugacy classes, keep track of based covering spaces $q : (E, e) \to (X, x_0)$ and covering isos preserving basepoints.

Galois? L/k finite Galois extn of fields then

$$\{E \mid k \subseteq E \subseteq L\} \cong \mathrm{Sub}(\mathrm{Gal}(L/k))$$ (similar, logically independent)

Pf Fix a universal cover $q : E \to X$ and $e_0 \in E$ with $q(e_0) = x_0$.

Then
$$\pi_1(X, x_0) \cong \mathrm{Aut}_q(E)$$
$$[\gamma] \longmapsto \underbrace{\varphi_\gamma : e_0 \mapsto e_0 \cdot \gamma}_{\substack{\text{unique covering auto} \\ \text{satisfying } \varphi_\gamma(e_0) = e_0 \cdot \gamma}}$$

Given $H \leq \pi_1(X, x_0)$ let
$$\hat{H} \leq \mathrm{Aut}_q(E)$$
denote the (isomorphic) image of H in $\mathrm{Aut}_q(E)$.

Have
$$\begin{tikzcd} E \arrow[r, "Q"] \arrow[d, "q"'] & \hat{E} := E/\hat{H} \arrow[dl, dashed, "\hat{q}"] \\ X & \end{tikzcd}$$

$\hat{q}$ exists, cts by univ property

WTS $\hat{q}$ is a covering map.

$$\hat{q}(e \cdot \hat{H}) = q(e)$$
$$= q(e\gamma), \quad \gamma \in H$$

For $U \subseteq X$ open, evenly covered, let $\hat{U}_0$ be a component of $\hat{q}^{-1} U$.

Suffices to show $\hat{q}|\hat{U}_0$ homeo. Have $Q^{-1}\hat{U}_0$ open & closed in $q^{-1}U \Rightarrow Q^{-1}\hat{U}_0$ is a union of components in $q^{-1}U$.

For U_0 a component of $Q^{-1}\hat{U}_0$, have

$$\begin{tikzcd} U_0 \arrow[rd, "Q"] \arrow[dd, "q", "\cong"'] & \\ & \hat{U}_0 \arrow[ld, "\hat{q}"] \\ U & \end{tikzcd}$$

so Q injective on U_0.

$Q\circ\varphi = Q$ for $\varphi\in\hat{H} \Rightarrow Q(\varphi U_0) = QU_0$ for $\varphi\in\hat{H}$.

Since Q is surj and $Q^{-1}\hat{U}_0 = \bigcup_{\varphi\in\hat{H}} \varphi U_0$, learn that $Q|_{U_0}$ is surj.

Thus $Q|_{U_0}$ is an open bij'n $\Rightarrow Q|_{U_0} : U_0 \cong \hat{U}_0$.

Hence $\hat{q}|\hat{U}_0 = (Q|_{U_0})\circ(q^{-1}|U)$ is a homeo as well.

Now check $\hat{q}_* \pi_1(\hat{E}, \hat{e}_0) = H$ for some $\hat{e}_0 \in \hat{E}$ s.t. $\hat{q}(\hat{e}_0) = x_0$.

Take $\hat{e}_0 = Q(e_0)$. Then $\hat{q}_* \pi_1(\hat{E}, \hat{e}_0) =$ isotropy of $\hat{e}_0$ under $\hat{E} \circlearrowleft \pi_1(X, x_0)$. For $[\gamma] \in \pi_1(X, x_0)$

$$\hat{e}_0 \cdot [\gamma] = Q(e_0) \cdot [\gamma] = Q(e_0 \cdot [\gamma]) = Q(\varphi_\gamma(e_0)).$$

$$\parallel$$

$$Q(e_0) \cdot [\gamma].$$

$Q : q^{-1} x_0 \to \hat{q}^{-1} x_0$

$\pi_1(X, x_0)$-equivariant

Thus $[\gamma]$ is isotropy iff $Q(\varphi_\gamma(e_0)) = Q(e_0)$

iff $\varphi_\gamma \in \hat{H}$

iff $\gamma \in H$.

$H \leq G$

$G \circlearrowright A \xrightarrow[Q]{} A/H$

$Q(ga) = Q(a) \iff g \in H$

$Q : E \longrightarrow E/\hat{H}$

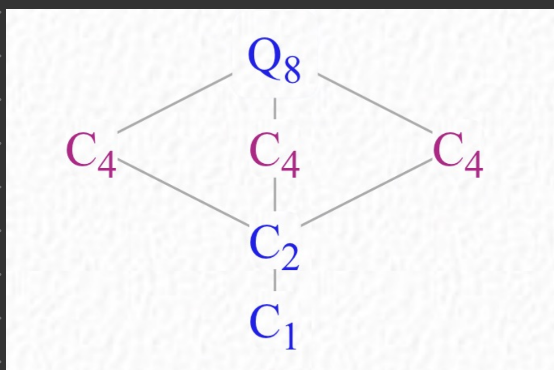
$\hat{H} = \{\varphi_\gamma \mid \gamma \in H\}$

This shows {covers} $\to$ {conj classes} is surjective.

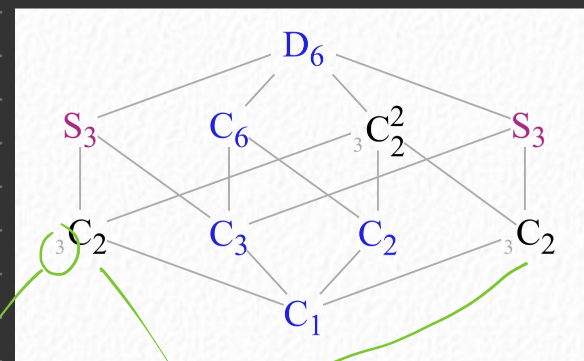
By Counting Isomorphism Criterion (11.40) injective as well. $\square$

For easy access to $\mathrm{Sub}(G)/G$ — conj action — so subgps up to conjugacy

see the GroupNames database.



\# conj subgps



etc.

different conj classes of $\cong$ subgps

$H \leq K \leq \pi_1(X, x_0)$

E univ cover

$$\begin{array}{c} E = E/\{1\} \\ \downarrow \\ E/\hat{H} \\ \downarrow \\ E/\hat{K} \\ \downarrow \\ X = E/\mathrm{Aut}_q(E) \end{array}$$

$\mathrm{Sub}(C_n)$

$\cong \{d \geq 1 \mid d \mid n\}$

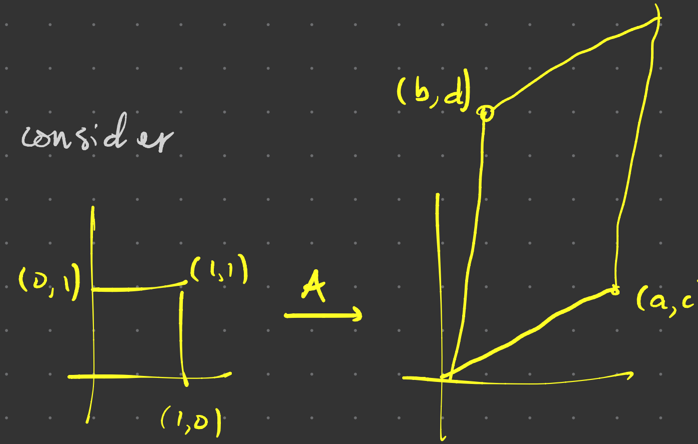
C_{p^i} : $\mathrm{Sub}(C_{p^n})$ $\quad$ p prime

$\cong$

i $\quad \{0 < 1 < \cdots < n\}$

E.g. Coverings of $\mathbb{T}^2$:

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R}) \cap \mathbb{Z}^{2\times 2}$, consider



$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "A"] \arrow[d, "\varepsilon_2"'] & \mathbb{R}^2 \arrow[d] \\ \mathbb{T}^2 \arrow[r, "q"'] & \mathbb{T}^2 \end{tikzcd}$$

$$(z,w) \longmapsto (z^a w^b, z^c w^d)$$

If $\ker(q) \leq \mathbb{T}^2$ is discrete, then q is a covering map.

$$\begin{tikzcd} A^{-1}\mathbb{Z}^2 \ni A^{-1}\begin{pmatrix} m \\ n \end{pmatrix} \arrow[r, mapsto] \arrow[d, mapsto] & (m,n) \in \mathbb{Z}^2 \arrow[d, mapsto] \\ (z,w) \arrow[r, mapsto] & (1,1) \end{tikzcd}$$

so $\ker q = \varepsilon_2 A^{-1}\mathbb{Z}^2 \leq \mathbb{T}^2$

torsion subgp gen'd by 2 elts

$\implies \ker q$ finite $\implies \ker q$ discrete.

Prop Every cover of $\mathbb{T}^2$ is isomorphic to precisely one of the following: (a) universal covering $\varepsilon_2 : \mathbb{R}^2 \longrightarrow \mathbb{T}^2$

(b) $q : S^1 \times \mathbb{R} \longrightarrow \mathbb{T}^2$

$$(z, y) \longmapsto \left(z^a \varepsilon(y)^b, \; z^b \varepsilon(y)^{-a}\right)$$

for $(a, b) \in \mathbb{N} \times \mathbb{Z}$ with $b > 0$ if $a = 0$

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

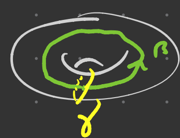
(c) $q : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$

$$(z, w) \longmapsto \left(z^a w^b, \; w^c\right)$$

with $\boxed{a > b \geq 0, \; c > 0 \text{ integers.}}$

Pf Fix $p = (1,1) \in \mathbb{T}^2$ as basepoint. Have $\pi_1(\mathbb{T}^2, p) \cong \langle \beta, \gamma \mid \beta\gamma = \gamma\beta \rangle$

$\cong \mathbb{Z}^2$.



<u>Fact</u> Subgps of $\mathbb{Z}^2$ are one of the following:

rank 0 — (i) trivial

rank 1 — (ii) infinite cyclic gen'd by (a,b) with $a \geq 0$, and $b > 0$ if $a = 0$

rank 2 — (iii) $\langle (a,0), (b,c) \rangle$ with $a > b \geq 0$, $c > 0$.

We check that $H \leq \mathbb{Z}^2$ free Abelian of rank 2 has type (iii).

Have $\underbrace{H \cap (\mathbb{Z} \times \{0\})}_{=: H_1} \neq \emptyset$ b/c $j(m,n) - n(i,j) = (jm - ni, 0) \in H \cap (\mathbb{Z} \times 0)$

basis for H

Take $\langle (a,0) \rangle = H_1$ w/ $a > 0$.

May "extend basis" to $(a,0)$, (b,c) satisfying (iii).

Given 2 such bases, $\exists M \in GL_2 \mathbb{Z}$ s.t. $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} M = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}$

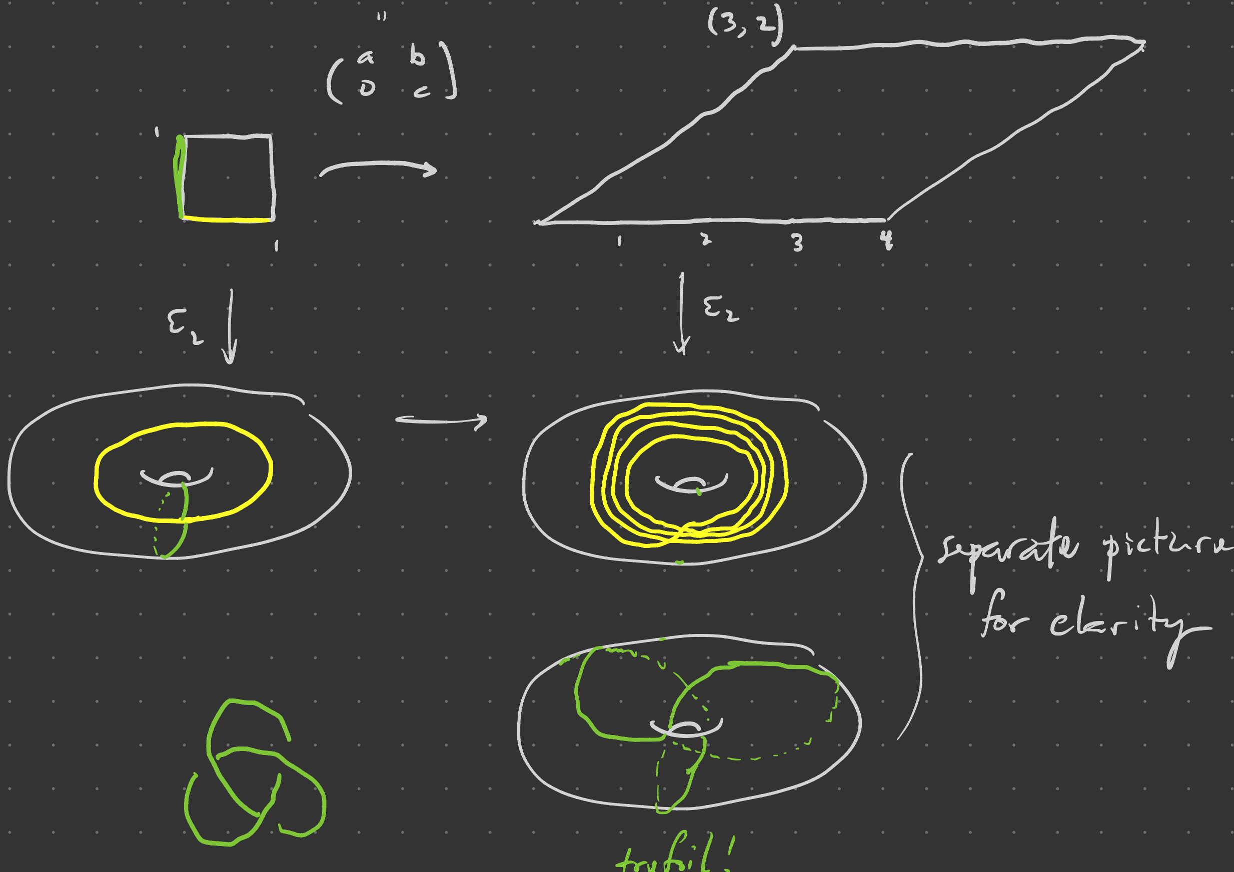
so M upper Δ'r with det 1. $\underset{\text{algebra}}{\rightsquigarrow}$ $M = \mathrm{id}$

so unique such basis!

Finally, check that induced subgps match. E.g. for (c)

$\beta \mapsto \beta^a$, $\gamma \mapsto \beta^b \gamma^c$

$$\begin{pmatrix} 4 & 3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

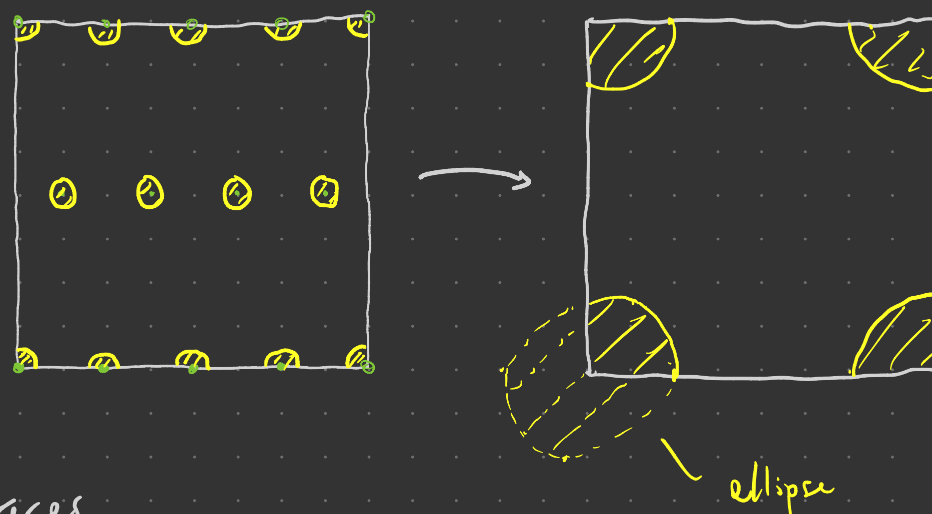


$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \implies A^{-1} = \begin{pmatrix} \frac{1}{a} & \frac{-b}{ac} \\ 0 & \frac{1}{c} \end{pmatrix}$$ so $A^{-1}\mathbb{Z}^2 \cap [0,1]^2$

looks like:

$$A = \begin{pmatrix} 4 & 3 \\ 0 & 2 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{4} & -3/8 \\ 0 & \frac{1}{2} \end{pmatrix}$$



E.g.: Lens spaces

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1\bar{z}_1 + z_2\bar{z}_2 = 1\}.$$

Fix $1 \le m < n$ rel prime integers.

$\mathbb{Z}/n \circlearrowright S^3$ by $[k]\cdot(z_1, z_2) = (e^{2\pi i k/n} z_1, \; e^{2\pi i k m/n} \;)$

cyclic order n

<u>Fact</u> $S^3 \xrightarrow[\text{cover}]{} \underbrace{S^3/(\mathbb{Z}/n)}_{\text{compact 3-mfld}} =: L(n,m)$

$$\pi_1(L(n,m)) \cong \mathbb{Z}/n \quad \text{since} \quad \pi_1 S^3 = 1$$

$$\mathrm{Sub}(\underbrace{\mathbb{Z}/n}_{\text{Abelian}}) = \left\{ r\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\left(\tfrac{n}{r}\right)\mathbb{Z} \;\middle|\; r \mid n \right\}$$

So $L(n,m)$ has one iso class of cover for each divisor of n.

∘ ∘ ∘ { Covers of $S^1 \vee S^1$