Automorphisms of a covering

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$q : E \to X$ covering

$$\mathrm{Aut}_q(E) := \left\{ \begin{tikzcd} E \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "q"'] & & E \arrow[dl, "q"] \\ & X & \end{tikzcd} \right\}$$

aka deck transformations

or covering tranformations

Prop $q : E \to X$ covering, $\varphi, \psi \in \mathrm{Aut}_q(E)$

(a) $\varphi = \psi \iff \exists e \in E$ s.t. $\varphi(e) = \psi(e)$

(b) $\varphi\big|_{q^{-1}\{x\}}$ is a $\pi_1(X, x)$-equivariant automorphism of $q^{-1}\{x\}$

(c) $U \subseteq X$ evenly covered open $\Rightarrow$ φ permutes components of $q^{-1}U$

(d) $\mathrm{Aut}_q(E) \circlearrowright E$ freely $\square$

E.g. $\mathrm{Aut}_{\varepsilon_n}(\mathbb{R}^n) \cong \mathbb{Z}^n$

$$\begin{array}{c} x \\ \downarrow \\ x+k \end{array} \longleftarrow\!\shortmid (k_1, \ldots, k_n) = k$$

E.g. $q: S^n \to \mathbb{RP}^n$ has $\mathrm{Aut}_q(S^n) = \{id, \text{antip}\}$.

Thm (orbit criterion for covering automorphisms) $q: E \to X$ covering. If $e_1, e_2 \in q^{-1}\{x\}$, then $\exists\, \varphi \in \mathrm{Aut}_q(E)$ s.t. $\varphi(e_1) = e_2$ iff $q_* \pi_1(E, e_1) = q_* \pi_1(E, e_2)$. $\square$

Cor $\mathrm{Aut}_q(E)$ acts transitively on each fiber iff q is normal (i.e. some/all $q_* \pi_1(E, e) \trianglelefteq \pi_1(X, x)$). $\square$

Thm $q: E \to X$ covering, $x \in X$. Then

$$\mathrm{Aut}_q(E) \xrightarrow{\cong} \underbrace{\mathrm{Aut}_{\pi_1(X,x)}(q^{-1}\{x\})}_{\pi_1(X,x)\text{-equivariant automorphisms of fiber}} = \{\eta : q^{-1}\{x\} \to q^{-1}\{x\} \mid \eta \ \pi_1\text{-eq bij}^n\}$$

iso of gps

$$\varphi \longmapsto \varphi|_{q^{-1}\{x\}}$$

$f: A \to B$ G-iso
$a \mapsto b$
$G_b = G_a$ $\quad ga = a$, $f(ga) = b$, $gf(a)$, gb

Pf Prop (b) $\Rightarrow$ homomorphism.
Prop (a) $\Rightarrow$ injective.
For surjective, suppose $\eta : q^{-1}\{x\} \xrightarrow{\cong} q^{-1}\{x\}$. For $e_1 \in q^{-1}\{x\}$, $e_2 = \eta(e_1)$, the isotropy of e_1, e_2 are the same; thus $q_* \pi_1(E, e_1) = q_* \pi_1(E, e_2) \Rightarrow \exists \varphi \in \mathrm{Aut}_q(E)$ with

$\varphi(e_1) = e_2$. Then $\eta, \varphi|_{q^{-1}\{x\}}$ are $\pi_1(X,x)$-equivariant isos of $q^{-1}\{x\}$ agreeing at e_1, so they are equal. $\square$

Thm (Covering Aut Group Structure) $q : E \longrightarrow X$ covering, $e \longmapsto x$

$G = \pi_1(X,x)$, $H = q_* \pi_1(E,e) \leq G$. For each $\gamma \in N_G(H)$ $\exists!\ \varphi_\gamma \in \mathrm{Aut}_q(E)$ s.t. $\varphi_\gamma(e) = e \cdot \gamma$. The map $N_G(H) \longrightarrow \mathrm{Aut}_q(E)$ is a surjective group hom with

$$\gamma \longmapsto \varphi_\gamma$$

kernel H $\implies$ $W_G(H) \cong \mathrm{Aut}_q(E)$.

i.e. $N_{\pi_1(X,x)}(q_* \pi_1(E,e)) / q_* \pi_1(E,e)$

Pf We have

$$
\begin{aligned}
W_G(H) &\xrightarrow{\;\cong\;} \mathrm{Aut}_G(q^{-1}\{x\}) \xleftarrow{\;\cong\;} \mathrm{Aut}_q(E). \\
H\gamma &\longmapsto \left(q^{-1}\{x\} \xrightarrow{\cong} q^{-1}\{x\} \right) \\
&\qquad\quad \text{s.t. } e \longmapsto e\cdot\gamma \\
&\qquad\quad \varphi\big|_{q^{-1}\{x\}} \longleftarrow\!\shortmid\ \varphi
\end{aligned}
$$

$$H\gamma \longmapsto \varphi_\gamma$$

$\square$

Cor (Normal case) If $q: E \to X$ is a normal covering, then $\forall x \in X$, $e \in q^{-1}\{x\}$,

$$
\begin{aligned}
\pi_1(X,x)/q_*\pi_1(E,e) &\xrightarrow{\;\cong\;} \mathrm{Aut}_q(E) \\
(q_*\pi_1(E,e))\gamma &\longmapsto \varphi_\gamma
\end{aligned}
$$

$\square$

Cor (Universal cover case) If $q: E \to X$ covering w/ E simply connected, then $\pi_1(X,x) \xrightarrow{\cong} \mathrm{Aut}_q(E)$

$\gamma \longmapsto \varphi_\gamma$

i.e. $\mathrm{Aut}_q(\tilde{X}) \cong \pi_1(X,x)$. $\square$

Quotients by group actions

$$\begin{array}{c} E \\ q \downarrow \\ X \end{array} \qquad q(e_1) = q(e_2) \iff \exists\, \varphi \in \mathrm{Aut}_q(E) \text{ s.t. } \varphi(e_1) \stackrel{!}{=} e_2$$

- $\mathrm{Aut}_q(E) \circlearrowright E$ freely by homeomorphisms
- $\mathrm{Aut}_q(E) \circlearrowright q^{-1}\{x\}$ transitively for q normal

$\Longrightarrow$ $X \cong E/\mathrm{Aut}_q(E)$ with $e_1 \sim e_2$ iff $e_2 = \varphi(e_1)$ for some $\varphi \in \mathrm{Aut}_q E$.

for q normal

Now suppose Γ a group $\circlearrowright E$.

$$\boxed{\text{When is } q : E \to E/\Gamma \text{ a covering map?}}$$

Note $E \to E/\Gamma$ necessarily normal since Γ acts transitively on fibers.

aka properly discontinuous

Defn $\Gamma \circlearrowright E$ is a covering space action if it acts by homeomorphisms (i.e. is cts) and every $e \in E$ has a nbhd U s.t.

$$\forall g \in \Gamma, \quad U \cap (g \cdot U) = \emptyset \text{ unless } g = 1. \quad (U \cap (gU) \neq \emptyset \text{ iff } g = 1)$$

Call the action effective when $g \cdot e = e \ \forall e \in E \implies g = 1$ (so only the unit of Γ acts trivially).

Thm (Covering space quotient theorem) E conn'd, locally path conn'd, $\Gamma \circlearrowright E$ effectively by homeos. Then the quotient $q : E \to E/\Gamma$ is a covering map iff the action is a covering space action.

In this case, q is normal and $\mathrm{Aut}_q(E) = \Gamma$.

Pf First assume q is covering. Each $g \in \Gamma$ induces a covering automorphism, so $\Gamma \leq \mathrm{Aut}_q E$. Check that $\mathrm{Aut}_q E \curvearrowright E$ is covering, and the restriction of a covering action to a subgroup is covering.

For the converse, suppose $\Gamma \curvearrowright E$ is covering. Have $q : E \to E/\Gamma$ cts, surjective, open. Take $x \in E/\Gamma$ and choose $e \in q^{-1}\{x\}$. Take U a nbhd of e s.t. $\forall g \in \Gamma$, $U \cap (g \cdot U) = \emptyset$ unless $g = 1$. WLOG, U path conn'd. Take $V = qU$, necessarily a path conn'd nbhd of x. Then $q^{-1}V = \coprod_{g \in \Gamma} g \cdot U$. Since

$$\begin{tikzcd} U \arrow[rr, "g", "\cong"'] \arrow[dr, "q"'] & & g \cdot U \arrow[dl] \\ & V & \end{tikzcd},$$

suffices to show $q|_U : U \xrightarrow{\cong} V$. $q|_U$ is surj, cts, open, and it is injective b/c $q(e) = q(e')$ for $e, e' \in U \Rightarrow e' = g \cdot e$ for some $g \in \Gamma$;

Since $\Gamma \circlearrowright E$ is covering, get $e = e'$. Thus q is a covering map. $\square$

Prop $\Gamma \leq G \Rightarrow G \circlearrowleft \Gamma$ is a covering space action $\Rightarrow q: G \to G/\Gamma$ is a normal covering map.

Γ: discrete subgp

G: conn'd top'l group

Thus $\Gamma \trianglelefteq \pi_1(G/\Gamma)$, not normal in G!

Pf Γ discrete $\Rightarrow \exists$ nbhd V of 1 in G s.t. $V \cap \Gamma = \{1\}$. Define $F: G \times G \to G$, $(g,h) \mapsto g^{-1}h$. Then $F^{-1}V$ is a nbhd of $(1,1) \Rightarrow \exists$ product nbhd $U_1 \times U_2$ s.t. $(1,1) \in U_1 \times U_2 \subseteq F^{-1}V$.

Set $U = U_1 \cap U_2$. Then $g, h \in U \Rightarrow g^{-1}h \in V$.

WTS $\mathcal{U}$ satisfies covering action condition: $\mathcal{U} \cap (\mathcal{U} \cdot g) \neq \emptyset \Rightarrow g = 1$.

Suppose $g \in \Gamma$ s.t. $\mathcal{U} \cap (\mathcal{U} \cdot g) \neq \emptyset$. Then $\exists h \in \mathcal{U}$ s.t. $hg \in \mathcal{U}$.

By construction, $g = h^{-1}(hg) \in V \cap \Gamma \Rightarrow g = 1$ as desired. $\square$

Cor Suppose G, H conn'd, loc path conn'd top'l gps, $\varphi: G \longrightarrow H$ surj cts homomorphism with discrete kernel. If φ is an open or closed map, then it is a normal covering. $\square$

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E.g.

$$\begin{array}{ccc} SU(2) & \longrightarrow & SO(3) \\ \cong & & \cong \\ S^3 & & \mathbb{RP}^3 \end{array}$$