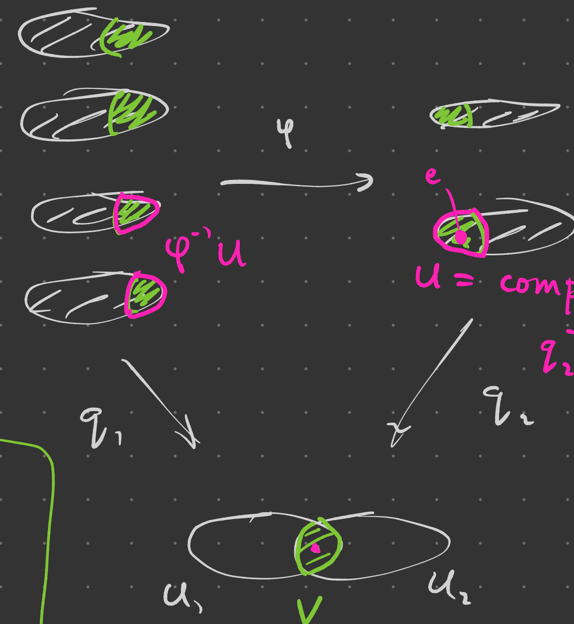
11.I.23

Evenly covered nbhds:

Check that for $e \in E_2$, U is evenly covered by $\varphi^{-1}U$. $\square$



U = component of $q_2^{-1}V$ containing e.

↳ might have to take conn'd component of $U_1 \cap U_2$ containing x

Thm

$$\begin{tikzcd} e_1 \arrow[rrrr, dashed, bend left] \arrow[ddrr, mapsto] & E_1 \arrow[rr, dashed, "\varphi"] \arrow[dr, "q_1"'] & & E_2 \arrow[dl, "q_2"] & e_2 \arrow[ddll, mapsto] \\ & & X & & \\ & & q_1(e_1) = q_2(e_2) & & \end{tikzcd}$$

$\exists$ covering hom $\varphi : e_1 \mapsto e_2$ iff $q_{1*}\pi_1(E_1, e_1) \leq q_{2*}\pi_1(E_2, e_2)$.

Pf Lifting criterion. $\square$

E.g.

$$\begin{tikzcd} S^1 \arrow[rr, dashed, "p_{m/n}"] \arrow[dr, "p_m"'] & & S^1 \arrow[dl, "p_n"] \\ & S^1 & \end{tikzcd} \iff n \mid m$$

"cyclotomic"

Thm (Covering Iso Criterion)

(a) In the previous setting, an iso $\varphi : e_1 \longmapsto e_2$ exists iff $q_{1*}\pi_1(E_1, e_1) = q_{2*}\pi_1(E_2, e_2)$.

necessarily unique!

(b) $q_1 \cong q_2$ iff for some (all) $x \in X$, the conjugacy classes of subgps of $\pi_1(X, x)$ induced by q_1, q_2 are the same.

(Recall Conj classes given by varying $e \in q_i^{-1}\{x\}$, applying q_{i*}.)

Pf (a) is formal given previous theorem.

(b) follows from isotropy analysis. $\square$

see p. 297

Universal Covering Space

Prop (Universality of simply conn'd coverings)

$$\begin{tikzcd} E \arrow[rr, dashed, "\exists Q"] \arrow[dr, "q"'] & & E' \arrow[dl, "q'"] \\ & X & \end{tikzcd}$$

simply conn'd

$\Rightarrow$ any two simply conn'd are isomorphic (uniquely so up to choice of basepoints).

$\square$

Call a simply conn'd covering space of X a universal cover $\tilde{X}$ of X. N.B. $\tilde{X}$ unique up to iso.

E.g. $\varepsilon_n : \mathbb{R}^n \longrightarrow \mathbb{T}^n$ exhibits $\mathbb{R}^n = \widetilde{\mathbb{T}}^n$.

The Cayley complex $\tilde{X}_G$ is a universal cover of the presentation complex X_G.

Call X locally simply conn'd when it admits a basis of simply conn'd open sets.

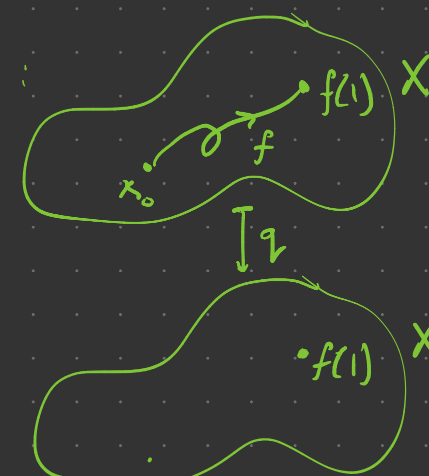
Thm Every conn'd locally simply conn'd space has a universal covering space.

path space of X (based at x_0)

Pf Fix $x_0 \in X$ and define $PX := \{[f: I \to X] \mid f(0) = x_0\}$

and $q: PX \longrightarrow X$
$\qquad [f] \longmapsto f(1)$.

Point in PX :



- Give PX the following topology: for $[f] \in PX$, $U \subseteq X$ open simply conn'd containing $f(1)$,

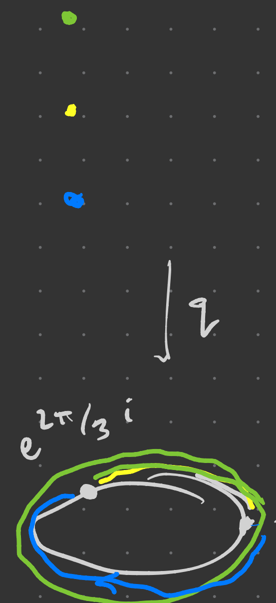
TPS Consider $X = S^1$. Does PX recover $\mathbb{R}$?

$$PX = \cancel{\{\cdot\} \times \mathbb{Z}}$$

$$\mathbb{R}$$

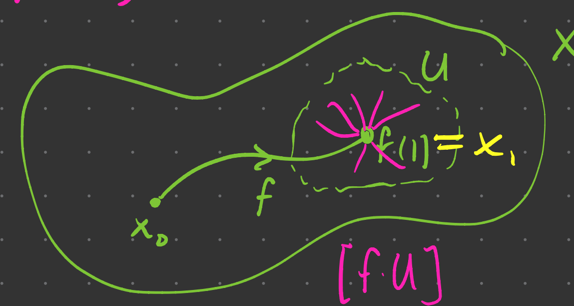
$[g \cdot u] \cong u$

$[f \cdot u] \cong u$



define $[f\cdot U] \subseteq PX$ by $[f\cdot U] := \{[f\cdot a] \mid a \text{ is a path in } U \text{ starting at } f(1)\}$

Then $\mathcal{B} = \{[f\cdot U]\}$ is a basis. (p.299)



- PX is path conn'd:

  Given $[f] \in PX$, define

  $$\begin{aligned} \tilde{f} : I &\longrightarrow PX \\ t &\longmapsto [f_t] \end{aligned} \quad \text{where} \quad \begin{aligned} f_t : I &\longrightarrow X \\ s &\longmapsto f(ts) \end{aligned}$$

  Then $\tilde{f}(0) = [c_{x_0}]$ and $\tilde{f}(1) = [f]$. Check $\tilde{f}$ is also cts.

- q is a covering map: For $U \subseteq X$ open simply conn'd,

  $$q^{-1}U = \{[f] \in PX \mid f(1) \in U\} = \coprod_{[f]} [f\cdot U]$$

  fix $x_1 \in U$. Varies over distinct

path classes $x_0 \rightsquigarrow x_1$.

Check q cts, homeo $[f \cdot U] \to U$ on each component

- PX is simply conn'd: Suppose $F : I \longrightarrow PX$ loop based at $[c_{x_0}]$.

F is a lift of $f := qF$. If $\tilde{f} : I \longrightarrow PX$, $t \longmapsto [f_t]$ then $q\tilde{f}(t) = q(f_t)$ $= f_t(1) = f(t)$ so $\tilde{f}$ lifts f stating at $[c_{x_0}]$. By unique lifting, $F = \tilde{f}$. Since F is a loop,

$$[c_{x_0}] = F(1) = \tilde{f}(1) = [f_1] = [f]$$

so f is nullhomotopic. By monodromy, F is as well. $\square$

$$\begin{tikzcd} & PX \arrow[d, "q"] \\ I \arrow[ur, "F"] \arrow[ur, bend left=60, "\tilde{f}"] \arrow[r, "f := qF"'] & X \end{tikzcd}$$