10. I. 23

Recall the lifting criterion:

$\tilde{\varphi}$ exists iff $\varphi_* \pi_1(Y, y_0) \leq q_* \pi_1(E, e_0)$

$$\begin{tikzcd} & E \arrow[d, "q"] \\ Y \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[r, "\varphi"'] & X \end{tikzcd}$$

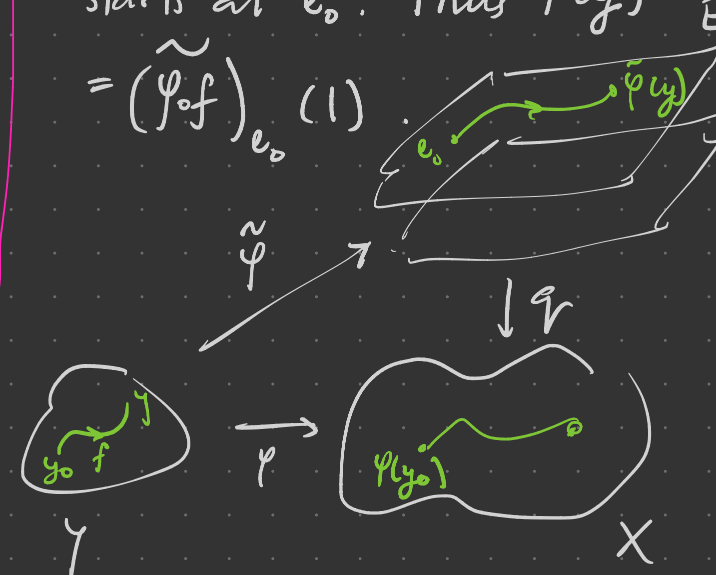
covering

cts

(conn'd, locally path conn'd

$y_0 \longmapsto \varphi(y_0) = q(e_0)$, $e_0 \mapsto q(e_0)$

Suppose $\tilde{\varphi}$ exists, $y \in Y$, $f : I \to Y$ w/ $f(0) = y_0$, $f(1) = y$. Then $\tilde{\varphi} \circ f$ lifts $\varphi \circ f$ and starts at e_0. Thus $\tilde{\varphi}(y) = \widetilde{(\varphi \circ f)}_{e_0}(1)$.



Idea: Define $\tilde{\varphi}(y) := \widetilde{(\varphi \circ f)}_{e_0}$.

To prove $\Leftarrow$ in lifting criterion, must show

(a) $\tilde{\varphi}$ is well-defined

(b) $\tilde{\varphi}$ is cts ~ read pp. 285-286

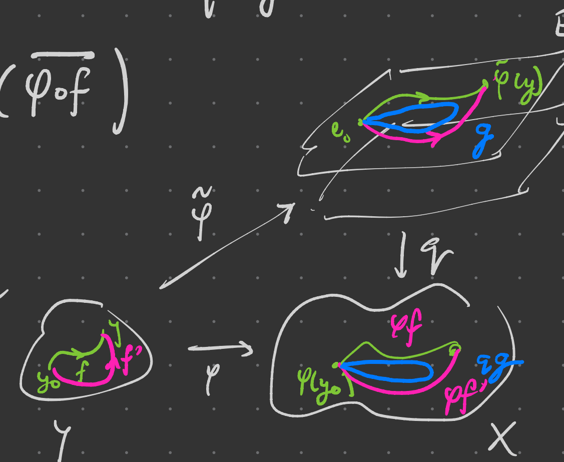
(a) Suppose f' is another path y_0 to y. Then $f' \cdot \bar{f}$ is a loop based at y_0, so $\varphi_*[f' \cdot \bar{f}] \in \varphi_* \pi_1(Y, y_0) \leq q_* \pi_1(E, e_0)$.

Thus $[\varphi \circ (f' \cdot \bar{f})] = [q \circ g]$ for some loop g in E based at e_0.

Hence $q \circ g \sim \varphi \circ (f' \cdot \bar{f}) \sim (\varphi \circ f') \cdot (\overline{\varphi \circ f})$

$\Rightarrow (q \circ g) \cdot (\varphi \circ f) \sim \varphi \circ f'$

By monodromy, the lifts of these have the same endpoint.



Since g lifts $q \circ g$, and g is a loop at e_0,

$$\left(\widetilde{\varphi \circ f'}\right)_{e_0}(1) = \left[g \cdot \left(\widetilde{\varphi \circ f}\right)_{e_0}\right](1) = \left(\widetilde{\varphi \circ f}\right)_{e_0}(1),$$

so the defn does not depend on the choice of f. $\square$

<u>Cor</u> (lifting from simply conn'd spaces) $q: E \longrightarrow X$ covering, Y simply conn'd locally path conn'd, then every cts $\varphi: Y \longrightarrow X$ has a lift to E. Give $y_0 \in Y$, the lift can be chosen to take y_0 to any pt in $q^{-1}\{\varphi(y_0)\}$.

<u>pf</u> $\varphi_* \pi_1(Y, y_0) = \{e\} \leq q_* \pi_1(E, e_0)$. $\square$

Cor (lifting maps to simply conn'd spaces) $q: E \to X$ covering, E simply conn'd. For any conn'd, loc path conn'd space Y, cts $\varphi: Y \to X$ lifts to E iff φ_* is the trivial hom for some $y_0 \in Y$. If this is the case, then the lift can be chosen to take y_0 to any elt of $q^{-1}\{\varphi(y_0)\}$.

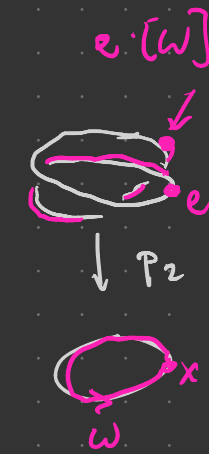
Q When can we solve the lifting problem

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "\varepsilon_2"] \\ \mathbb{RP}^2 \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[r, "\varphi"'] & \mathbb{T}^2 \end{tikzcd} \quad ? \qquad \overset{\pi_1}{\rightsquigarrow} \qquad \begin{tikzcd} & e \arrow[d] \\ \mathbb{Z}/2\mathbb{Z} \arrow[ur] \arrow[r] & \mathbb{Z}\times\mathbb{Z} \end{tikzcd}$$

always commutes so all φ lift.

Monodromy action

Thm (monodromy action) $q : E \to X$ covering, $x \in X$.
There is a transitive right action $q^{-1}\{x\} \circlearrowleft \pi_1(X,x)$
given by $e \cdot [f] = \tilde{f}_e(1)$.



$(e\cdot[f])\cdot[g] = e\cdot([f][g])$

Pf Read p. 287 for well-definition.
Transitivity: since E is path conn'd any two pts $e, e' \in q^{-1}\{x\}$
are joined by a path h in E. Set $f = q \circ h$ to see that
h is a lift of f starting at e, whence $e \cdot [f] = e'$. □

Algebraic interlude on G-sets

S a right G-set

For $s \in S$, the isotropy (aka stabilizer) group of s is

$$G_s := \{ g \in G \mid s \cdot g = s \}$$

Moral exc Check that $G_s \leq G$.

equivariant bijection

Thm (orbit-stabilizer) For $s \in S$, $sG \cong {}_{G_s}\backslash G$

G-orbit of s

right cosets $\{ G_s g \mid g \in G \}$

$$s \cdot g \longmapsto G_s g$$

GSet : right G-sets

$$\mathrm{GSet}(S,T) = \{ f : S \to T \mid \underbrace{f(s\cdot g) = f(s)\cdot g}_{\text{equivariance}} \ \forall g \in G \}$$

Every G-set is a disjoint union of orbits $\cong \coprod H_i \backslash G$

Prop $G_{s\cdot g} = g^{-1} G_s g$ so the set of isotropy groups for a transitive G-set; transitive G-sets S, T are isomorphic iff G_s conjugate to G_t for some (/all) $s \in S$, $t \in T$.

$$h \in G_s \implies (s\cdot g)(g^{-1} h g) = s \cdot (\cancel{g g^{-1}} h g) = (s\cdot h)\cdot g = s\cdot g$$

The Weyl group of $H \leq G$ is $W_G(H) := H \backslash \underbrace{N_G(H)}$

normalizer of H:
maximal subgroup containing H
in which H is normal.
$= \{ g \in G \mid gHg^{-1} = H \}$

Thm For S a transitive G-set, s_0 some elt of S,

$$\mathrm{Aut}_G(S) \cong W_G(G_{s_0})$$
$$\varphi_\gamma \longleftarrow G_{s_0}\gamma$$

unique G-equivariant map $S \longrightarrow S$
$s_0 \longmapsto s_0 \cdot \gamma$

Q Why does φ_γ exist and is well-defined?

Back to our favorite transitive G-set, the monodromy action

$$q^{-1}\{x\} \circlearrowleft \pi_1(X,x).$$

<u>Thm</u> (Isotropy groups for monodromy) $q : E \to X$ covering, $x \in X$. $\forall e \in q^{-1}\{x\}$, $\underbrace{\pi_1(X,x)_e}_{\text{isotropy gp of } e:\ \{g \in \pi_1(X,x) \mid e \cdot g = e\}} = q_* \pi_1(E,e) \leq \pi_1(X,x)$

<u>Pf</u> Let $e \in q^{-1}\{x\}$ be arbitrary and suppose $[f] \in \pi_1(X,x)_e$.

Then $\tilde{f}_e(1) = e \cdot [f] = e$, so $[\tilde{f}_e] \in \pi_1(E,e)$. We have

$q_*[\tilde{f}_e] = [q \circ \tilde{f}_e] = [f]$, so $[f] \in q_* \pi_1(E,e)$. Thus

$\pi_1(X,x)_e \subseteq q_* \pi_1(E,e)$.

For the opposite inclusion, if $[f] \in q_* \pi_1(E,e)$, then $\exists g : I \to E$ based at e s.t. $q_*[g] = [f] \Rightarrow q \circ g \sim f$.

Then $g = \widetilde{(q \circ g)}_e$ by uniqueness of lifts, and

$e \cdot [f] = e \cdot [q \circ g] = \widetilde{(q \circ g)}_e(1) = g(1) = e$ so $[f] \in \pi_1(X,x)_e$. $\square$

Cor $q : E \to X$ covering. The monodromy action is free on each fiber of q iff E is simply conn'd.

Pf The action is free on $q^{-1}\{x\}$ iff $q_* \pi_1(E,e)$ triv for any (all) $e \in q^{-1}\{x\} \iff \pi_1(E,e)$ triv since q_* injective. $\square$

In this case, $\underset{\text{G-set}}{q^{-1}\{x\} \cong G}$ for $G = \pi_1(X,x)$

Cor Suppose $q: E \longrightarrow X$ covering, E is simply conn'd. Then each fiber of q has cardinality $|\pi_1(X,x)|$.

Pf The monodromy action is free and $q^{-1}\{x\}$ is a free transitive $\pi_1(X,x)$-set. $\square$

E.g. $q: S^n \longrightarrow \mathbb{RP}^n$ covering w/ S^n simply conn'd for $n \geq 2$

$\implies \pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$.

Read p. 293 on normal coverings. $q_* \pi_1(E,e) \trianglelefteq \pi_1(X,x)$

for some/all e, x.

Can do same trick if $|q^{-1}(x)| = p$.

Happens for lens spaces.

Covering homomorphisms

$q_1 : E_1 \to X$, $q_2 : E_2 \to X$ coverings

A covering homomorphism is a cts map $\varphi : E_1 \to E_2$ s.t.

$q_2 \varphi = q_1$:

$$\begin{tikzcd} E_1 \arrow[rr, "\varphi"] \arrow[dr, "q_1"'] & & E_2 \arrow[dl, "q_2"] \\ & X & \end{tikzcd}$$

commutes.

If φ is also a homeo, call it a covering isomorphism.

Compare with field hom over k

$$\begin{tikzcd} L_1 \arrow[rr] \arrow[dr, no head] & & L_2 \arrow[dl, no head] \\ & k & \end{tikzcd}$$

Prop $q_i : E_i \to X$ coverings, $i = 1, 2$,

$$\begin{tikzcd} E_1 \arrow[rr, bend left, "\varphi"] \arrow[rr, "\psi"'] \arrow[dr, "q_1"'] & & E_2 \arrow[dl, "q_2"] \\ & X & \end{tikzcd}$$

(a) $\varphi = \psi$ iff $\exists e \in E_1$ s.t. $\varphi(e) = \psi(e)$

(b) $\forall x \in X$, $\varphi|_{q_1^{-1}\{x\}} : q_1^{-1}\{x\} \longrightarrow q_2^{-1}\{x\}$ is $\pi_1(X, x)$-equivariant

(c) Every covering hom is itself a covering map.

Pf (a) follows from unique lifting:

$$\begin{tikzcd} & E_2 \arrow[d, "q_2"] \\ E_1 \arrow[ur, shift left, "\varphi"] \arrow[ur, shift right, "\psi"'] \arrow[r, "q_1"'] & X \end{tikzcd}$$

(b) $\varphi(e \cdot [f]) = \varphi(\tilde{f}_e(1)) = (\varphi \circ \tilde{f}_e)(1) = \varphi(e) \cdot [f]$.

Q Justify this.

(c) Surjective: for $e \in E_2$, take $e_0 \in q_1^{-1}\{q_2(e)\} \neq \emptyset$. Then $\varphi(e_0) = e$.