6.I.23

Covering maps

Goal Emulate properties of $\varepsilon : \mathbb{R} \longrightarrow S^1$ in order to compute more fundamental groups.

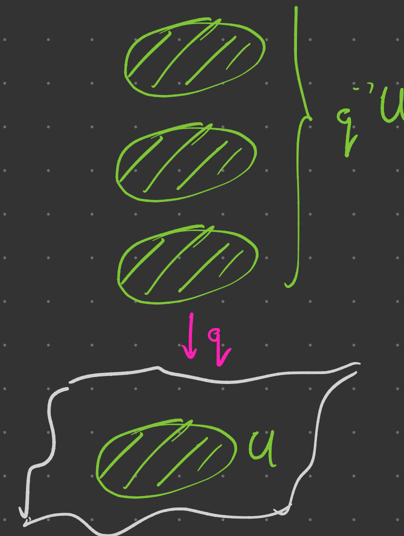
Recall f, g based loops in S^1 are path homotopic iff they have the same winding number $\tilde{f}(1) - \tilde{f}(0) = \tilde{g}(1) - \tilde{g}(0)$ for $\tilde{f}, \tilde{g}$ lifts of f, g along ε:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\varepsilon"] \\ I \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

Unique lifting, htpy lifting, path lifting, ...

Defn For E, X spaces, $q: E \to X$ cts, an open $U \subseteq X$ is **evenly covered** by q when $q^{-1}U$ is a disjoint union of conn'd open sets, each mapped homeomorphically onto U by q.

A **covering map** is a cts surj map $q: E \to X$ with E conn'd, locally path conn'd, and every pt of X has an evenly covered nbhd.



Eg.
$$\varepsilon: \mathbb{R} \longrightarrow S^1$$
$$t \longmapsto \exp(2\pi i t)$$

E.g. $p_n : S^1 \longrightarrow S^1$

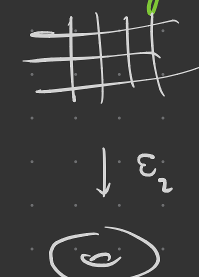
$z \longmapsto z^n$

Non-e.g. $\varepsilon|_{(0,2)}$

no evenly covered nbhd

E.g. $\varepsilon_n : \mathbb{R}^n \longrightarrow \mathbb{T}^n$

$(t_1, \dots, t_n) \longmapsto (\varepsilon(t_1), \dots, \varepsilon(t_n))$



E.g. $S^n \longrightarrow \mathbb{RP}^n$ 2-sheeted cover

$x \longmapsto$ line spanned by $x \subseteq \mathbb{R}^{n+1}$

Lifting Properties

A lift of $\varphi: Y \longrightarrow X$ along q is $\tilde{\varphi}: Y \longrightarrow E$ s.t. $q\tilde{\varphi} = \varphi$

i.e.

$$\begin{tikzcd} & E \arrow[d, "q"] \\ Y \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[r, "\varphi"'] & X \end{tikzcd}$$

Thm (Unique lifting) Let $q: E \longrightarrow X$ be a covering map. Suppose Y is conn'd, $\varphi: Y \longrightarrow X$ cts, $\tilde{\varphi}_1, \tilde{\varphi}_2 : Y \longrightarrow E$ are lifts of φ that agree at some point of Y. Then $\tilde{\varphi}_1 = \tilde{\varphi}_2$.

Pf Same as for ε. $\square$

Thm (Homotopy lifting) Let $q: E \longrightarrow X$ be a covering map, Y locally conn'd.
Suppose $\varphi_0, \varphi_1 : Y \longrightarrow X$ cts, $H: Y \times I \longrightarrow X$ a htpy from φ_0 to φ_1, $\tilde{\varphi}_0 : Y \longrightarrow E$ any lift of φ_0. Then $\exists!$ lift of H to $\tilde{H}$ with $\tilde{H}(-,0) = \tilde{\varphi}_0$. If H is stationary on some $A \subseteq Y$ then so is $\tilde{H}$.

$$\begin{tikzcd}
Y \times 0 \arrow[r, "\tilde{\varphi}_0"] \arrow[d] \arrow[dr, "\varphi_0"] & E \arrow[d, "q"] \\
Y \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & X \\
Y \times 1 \arrow[u] \arrow[ur, "\varphi_1"'] &
\end{tikzcd}$$

$\tilde{H} : \tilde{\varphi}_0 \simeq \tilde{H}(-,1)$

a lift of φ_1

Pf Same as for ε. $\square$

Cor (Path lifting) $q: E \to X$ covering, $f: I \to X$ a path, $e \in q^{-1}f(0) \subseteq E$. Then $\exists!$ lift $\tilde{f}: I \to E$ of f with $\tilde{f}(0) = e$.

Notation: $\tilde{f}_e$

Pf Ditto. $\square$

Winding number?

Thm (Monodromy) $q: E \to X$ covering map, $f, g: I \to X$ paths from p to q, $\tilde{f}_e, \tilde{g}_e$ lifts with same initial point e.

(a) $\tilde{f}_e \sim \tilde{g}_e$ iff $f \sim g$

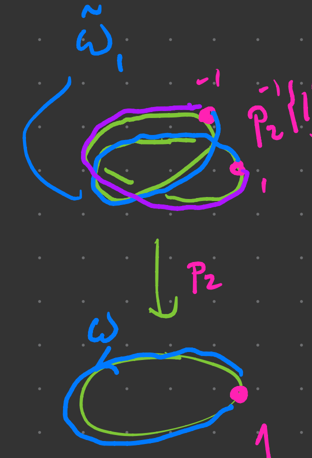
(b) If $f \sim g$, then $\tilde{f}_e(1) = \tilde{g}_e(1)$

converse holds for ε b/c $\mathbb{R}$ is simply conn'd.

Pf (a) If $\tilde{f}_e \sim \tilde{g}_e$, then composing w/ q witnesses $f \sim g$.

For the converse, suppose $H : f \sim g$. By htpy lifting, get $\tilde{H} : \tilde{f}_e \sim$ some lift of g starting at e. By unique lifting, this is just $\tilde{g}_e$.

(b) $f \sim g \Rightarrow \tilde{f}_e \sim \tilde{g}_e \Longrightarrow \tilde{f}_e(1) = \tilde{g}_e(1)$. $\square$



$[\omega] \cdot 1 = -1$

$[\omega] \cdot (-1) = 1$

Upshot $\pi_1(X, x) \curvearrowright q^{-1}\{x\}$ "monodromy action"

$$[f] \cdot e = \tilde{f}_e(1)$$

Thm (Injectivity) $q : E \to X$ covering. $\forall e \in E$,

$$q_* : \pi_1(E, e) \longrightarrow \pi_1(X, q(e)) \text{ is injective.}$$
$$[f] \longmapsto [q \circ f]$$

Pf. Suppose $[f] \in \ker(q_*)$ so $q_*[f] = [c_{q(e)}]$.

Then $qf \sim c_{q(e)}$ in X. By the monodromy theorem, any lifts of qf, $c_{q(e)}$ starting at the same point are path htpic in E.

$$\begin{tikzcd} & E \arrow[d, "q"] \\ I \arrow[ur, "f"] \arrow[r, "qf"'] & X \end{tikzcd}$$

so $f = \widetilde{qf}_e$ and c_e lifts $c_{q(e)}$.

These both start at e, so $f \sim c_e$, i.e. $[f]$ is trivial. $\square$

Upshot
$$\left\{ \text{coverings } \begin{matrix} E \\ \downarrow q \\ X \end{matrix} \right\} \longrightarrow \mathrm{Sub}(\pi_1(X, q(e)))$$
$$q \longmapsto \mathrm{im}(q_*) \leq \pi_1(X, q(e))$$
assigns subgroups of π_1 to coverings.

On top, we will solve the lifting problem:

Thm $\begin{array}{c} E \\ q \downarrow \\ X \end{array}$ covering, Y conn'd loc path conn'd, $\varphi : Y \longrightarrow X$ cts.

Given $y_0 \in Y$, $e_0 \in E$ with $q(e_0) = \varphi(y_0)$, φ has a lift $\tilde{\varphi} : Y \longrightarrow E$ s.t. $\tilde{\varphi}(y_0) = e_0$ iff $\varphi_* \pi_1(Y, y_0) \subseteq q_* \pi_1(E, e_0)$.

Pf of $\Rightarrow$

$$\begin{tikzcd} & \pi_1(E, e_0) \arrow[d, "q_*"] \\ \pi_1(Y, y_0) \arrow[ur, "\tilde{\varphi}_*"] \arrow[r, "\varphi_*"'] & \pi_1(X, \varphi(y_0)) \end{tikzcd}$$

$\square$

$$\begin{tikzcd}
 & E \arrow[d, "q"] & e_0 \arrow[dd, mapsto] \\
Y \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[r, "\varphi"'] & X & \\
y_0 \arrow[rr, mapsto] \arrow[uurr, bend left, mapsto] & & \varphi(y_0) = q(e_0)
\end{tikzcd}$$

exists <u>iff</u>

$$\varphi_* \pi_1(Y, y_0) \leq q_* \pi_1(E, e_0)$$

"Stairs"

upstairs

$$\begin{tikzcd}
E \arrow[d, "q"'] \\
X
\end{tikzcd}$$

down-stairs