Math 545:

Geometry & Topology of Manifolds

Winter 2023

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Presentation complex

Group $\quad G = \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$
$\quad = \langle S \mid R \rangle$

$G = \mathbb{Z} \times \mathbb{Z} \cong \langle x, y \mid xyx^{-1}y^{-1} \rangle$

2-dim cell cpx $\quad X_G$:

$(X_G)_0 = *$

$(X_G)_1 = \bigvee_S S^1$ (loops based at $*$ indexed by S)

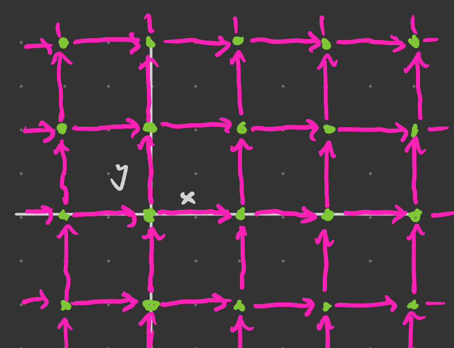
$(X_G)_2 =$ one 2-cell for each elt of R,
∂D^2 glued according to rel'n word

Cayley graph (connect-the-dots, grad school version)

$G = \langle S \mid R \rangle$

$\Gamma_G = \Gamma_{G,S} :=$ directed graph w/ vertices G,
edges $g \xrightarrow{x} gx$ for $x \in S$

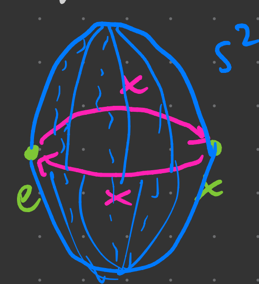
E.g. $G = \langle x, y \mid xyx^{-1}y^{-1} \rangle \cong \mathbb{Z}^2$

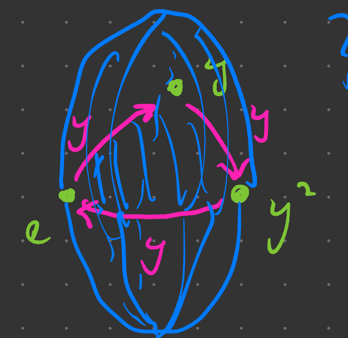


Q Given $g \in G$, $r = xyz\cdots \in R$, what happens if you follow the path from g using r as instructions?

P1 Draw Cayley graphs for

$C_2 = \langle x \mid x^2 \rangle$

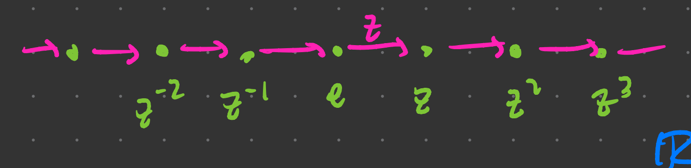


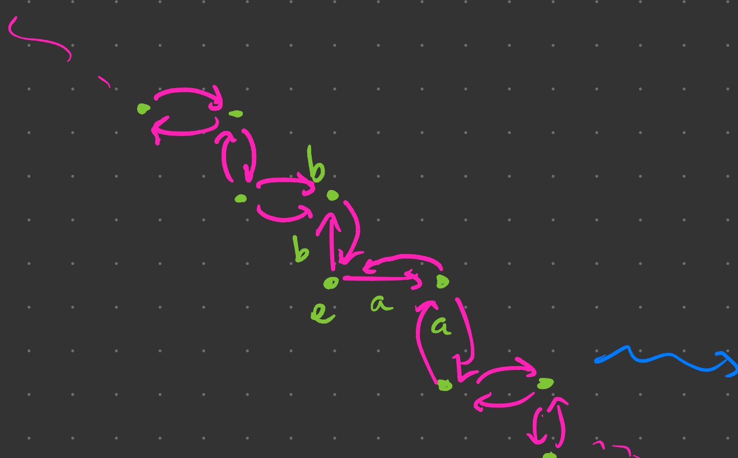


$C_3 = \langle y \mid y^3 \rangle$

$C_\infty = \langle z \rangle \cong \mathbb{Z}$

$C_2 * C_2 = \langle a, b \mid a^2, b^2 \rangle$





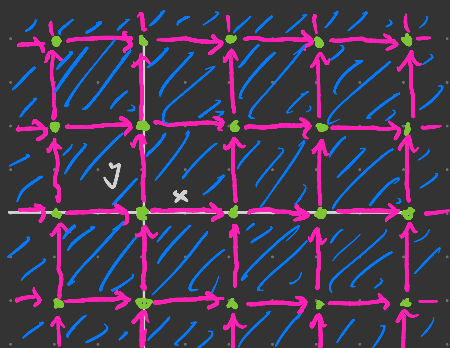
P2 Corresponding graphs?

Cayley Complex

$(\tilde{X}_G)_1 = \Gamma_G$

$(\tilde{X}_G)_2$: For each $g \in G$, $r \in R$, attach a 2-cell to Γ_G via the loop r starting at g.

E.g. $\tilde{X}_{\mathbb{Z}^2} \cong \mathbb{R}^2$



$G \curvearrowright \tilde{X}_G$ simply ~~transitively~~

$\forall x, y \; \exists g$ s.t. $gx = y$

unique!

(Equiv. free and transitive.)

Action $G \circlearrowright \tilde{X}_G$

Given $h \in G$, define action cell-wise (and check compatibility)

0-cells : $h \cdot g = hg$

1-cells : $h \cdot (g \xrightarrow{x} gx) = hg \xrightarrow{x} hgx$

2-cells : Take 2-cells to 2-cells homeomorphically w/ ∂ transformed as for 1-cells :

$r = xyzw$,

$$h \cdot \left(\begin{tikzcd} & gx \arrow[r, "y"] & gxy \arrow[dl, "z"] \\ g \arrow[ur, "x"] & gxyz \arrow[l, "w"] \end{tikzcd} \right) = \begin{tikzcd} & hgx \arrow[r, "y"] & hgxy \arrow[dl, "z"] \\ hg \arrow[ur, "x"] & hgxyz \arrow[l, "w"] \end{tikzcd}$$

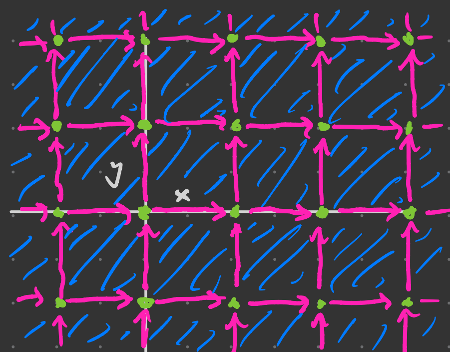
Check that this is a cts ~~simply transitive~~ action.
transitive on 0-cells

Quotient

Claim $\quad \tilde{X}_G / G \cong X_G$

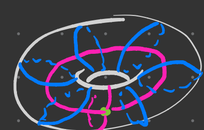
E.g.

$\mathbb{Z}^2 \circlearrowright$



$\tilde{X}_{\mathbb{Z}^2}$

$\downarrow$

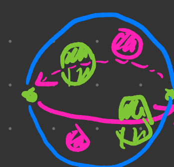


$X_{\mathbb{Z}^2}$

- Nice properties of $\tilde{X}_G \longrightarrow X_G$?

- Comparison with
  $$\mathbb{R} \longrightarrow S^1$$
  $$t \longmapsto \exp(2\pi i t) \quad ?$$

$G = C_2$



$\tilde{X}_{C_2}$

$\downarrow$

$\mathbb{RP}^2 \qquad X_{C_2}$

 $/$ antip on ∂

Goal 1 "Galois theory" of covering maps and π_1

$$\{\text{subgroups of } \pi_1 X\} \xleftrightarrow{\cong} \{\text{"covers" of } X\}$$

Note
- $\tilde{X}_G$ is a "universal cover" of X_G, corresponding to $e \leq \pi_1 X_G \cong G$
- Cover for $H \leq G$ arises as $\tilde{X}_G / H \longrightarrow X_G$.