MATH 545: MANIFOLDS
HOMEWORK DUE FRIDAY WEEK 9

Problems taken from *Introduction to Smooth Manifolds* are marked ISM x–y. Please review the syllabus for expectations and policies regarding homework.

Problem 1 (ISM 5–10). For each $a \in \mathbb{R}$, let M_a be the subset of $\mathbb{R}^2$ defined by

$$M_a := \{(x, y) \mid y^2 = x(x-1)(x-a)\}.$$

For which values of a is M_a an embedded submanifold of $\mathbb{R}^2$? For which values can M_a be given a topology and smooth structure making it into an immersed submanifold?

Problem 2 (ISM 5–6). Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, and let $UM \subseteq T\mathbb{R}^n$ be the set of unit tangent vectors to M:

$$UM := \{(x, v) \in T\mathbb{R}^n \mid x \in M, v \in T_x M, |v| = 1\}.$$

Prove that UM is an embedded $(2m-1)$-dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$.

Problem 3 (ISM 5–14). Prove ISM Theorem 5.32 (uniqueness of the smooth structure on an immersed submanifold once the topology is given).

Problem 4. Recall from the first day of Math 544 that the space of (labeled) equilateral n-gons is

$$M_n = \{(z_1, \ldots, z_{n-1}) \in \mathbb{T}^{n-1} \mid 1 + z_1 + \cdots + z_{n-1} = 0\}.$$

Prove that M_n is a codimension 2 embedded smooth submanifold of $\mathbb{T}^{n-1}$ for $n \geq 3$ odd. (*Bonus*: Prove that M_n is not smooth for $n \geq 4$ even.)

Problem 5 (ISM 6–2). Suppose M is an embedded n-dimensional submanifold of $\mathbb{R}^{2n+1}$. Let $UM \subseteq T\mathbb{R}^{2n+1}$ be the unit tangent bundle of M, and let $G\colon UM \to \mathbb{RP}^{2n}$ be the map $G(x, v) = [v]$. Use Sard's theorem to conclude that there is some $v \in \mathbb{R}^{2n+1} \smallsetminus \mathbb{R}^{2n}$ such that $[v]$ is not in the image of G, and show that the projection from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n}$ with kernel $\mathbb{R}v$ restricts to an immersion of M into $\mathbb{R}^{2n}$. (This proves the Whitney immersion theorem assuming the Whitney embedding theorem.)