MATH 545: MANIFOLDS
HOMEWORK DUE FRIDAY WEEK 7

Problems taken from *Introduction to Smooth Manifolds* are marked ISM x–y. Please review the syllabus for expectations and policies regarding homework.

Problem 1 (ISM 2–14). Suppose A and B are disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \leq f(x) \leq 1$ for all $x \in M$, $f^{-1}\{0\} = A$, and $f^{-1}\{1\} = B$.

Problem 2 (ISM 3–2). Prove Proposition 3.14: Let $M_1, \ldots, M_k$ be smooth manifolds, and for each j, let $\pi_j \colon M_1 \times \cdots \times M_k \to M_j$ be the projection onto the M_j factor. For any point $p = (p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$, the map

$$\begin{aligned} \alpha \colon T_p(M_1 \times \cdots \times M_k) &\longrightarrow T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k \\ v &\longmapsto (d(\pi_1)_p(v), \ldots, d(\pi_k)_p(v)) \end{aligned}$$

is a linear isomorphism. (The same is true if one of the spaces M_i is a smooth manifold with boundary, but you are not asked to prove that case.)

Problem 3 (ISM 3–4). Show that TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$.

Problem 4 (ISM 3–5). Let $S^1 \subseteq \mathbb{R}^2$ be the unit circle, and let $K \subseteq \mathbb{R}^2$ be the boundary of the square of side length 2 centered at the origin. Show that there is a homeomorphism $F \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(S^1) = K$, but there is no *diffeomorphism* with the same property. [*Hint*: Let γ be a smooth curve whose image lies in S^1, and consider the action of $dF(\gamma'(t))$ on the coordinate functions x and y.] Why does this prove that the Cartesian product of manifolds with boundary is not necessarily a manifold with boundary?

Problem 5 (ISM 4–2). Suppose M is a smooth manifold (without boundary), N is a smooth manifold with boundary, and $F \colon M \to N$ is smooth. Show that if $p \in M$ is a point such that dF_p is nonsingular, then $F(p) \in N^\circ$.