# MATH 545: MANIFOLDS
# HOMEWORK DUE FRIDAY WEEK 5

Problems taken from *Introduction to Topological Manifolds* are marked ITM $x$–$y$; problems taken from *Introduction to Smooth Manifolds* are marked ISM $x$–$y$. Please review the syllabus for expectations and policies regarding homework.

*Problem* 1 (ITM 13–7, BROUWER FIXED POINT THEOREM). For each integer $n \geq 0$, prove that every continuous map $f\colon \overline{\mathbb{B}}^n \to \overline{\mathbb{B}}^n$ has a fixed point. (Use homology to emulate the fundamental group-based proof for $n = 2$.)

*Problem* 2 (ITM 13–8). Show that if $n$ is even, then $C_2 = \{\pm 1\}$ is the only nontrivial group that can act freely and continuously on $S^n$. (*Hint*: Show that if $G$ acts continuously on $S^n$, then degree induces a homomorphism $G \to C_2$. *Bonus*: What happens when $n$ is odd?)

*Problem* 3 (ITM 13–9). Use the cell structure of $\mathbb{RP}^3$ to compute its homology. (*Bonus*: Compute the homology of $\mathbb{RP}^n$ for all $n$.)

*Problem* 4 (ISM 1–8). An *angle function* on a subset $U \subseteq S^1 \subseteq \mathbb{C}$ is a continuous function $\theta\colon U \to \mathbb{R}$ such that $\exp(i\theta(z)) = z$ for all $z \in U$. Show that there exists an angle function $\theta$ on an open subset $U \subseteq S^1$ if and only if $U \neq S^1$. For any such angle function, show that $(U, \theta)$ is a smooth coordinate chart for $S^1$ with a standard smooth structure.

*Problem* 5 (ISM 1–9). For $n \geq 0$, *complex projective space*, $\mathbb{CP}^n$, is the set of all 1-dimensional $\mathbb{C}$-linear subspaces of $\mathbb{C}^{n+1}$ with the quotient topology induced by the natural projection $\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$. Show that $\mathbb{CP}^n$ is a compact $2n$-dimensional topological manifold, and show how to give it a smooth structure analogous to the smooth structure on $\mathbb{RP}^n$.