MATH 545: MANIFOLDS
HOMEWORK DUE FRIDAY WEEK 4

Problems taken from *Introduction to Topological Manifolds* are marked ITM x–y. Please review the syllabus for expectations and policies regarding homework.

Problem 1 (12–4). Let $\mathscr{E} \cong S^1 \vee S^1$ be the figure-eight space consisting of unit circles centered at $\pm i$ in $\mathbb{C}$, and let X be the union of the real axis of $\mathbb{C}$ with infinitely many unit circles centered at $2\pi k + i$, $k \in \mathbb{Z}$. Let $q\colon X \to \mathscr{E}$ be the map sending each circle in X onto the upper circle in $\mathscr{E}$ by translation by a real number, and sending the real axis onto the lower circle by $x \mapsto ie^{ix} - i$. You may assume that q is a covering map.

(a) Identify the subgroup $q_*\pi_1(X, 0)$ of $\pi_1(\mathscr{E}, 0)$ in terms of the generators of $\pi_1(\mathscr{E}, 0)$.
(b) Determine the automorphism group $\operatorname{Aut}_q(X)$.
(c) Determine whether q is a normal covering.

Problem 2 (12–6). Let

$$E = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid x \neq y\}$$

considered as a subspace of $\mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$. Define an equivalence relation $\sim$ on E by $(x, y) \sim (y, x)$ for all $(x, y) \in E$. Compute the fundamental group of $E/{\sim}$. (*Note*: The space $E/{\sim}$ is the *unordered configuration space of two points in* $\mathbb{R}^3$.)

Problem 3 (12–9). Find a covering space action of a group Γ on the plane such that $\mathbb{R}^2/\Gamma$ is homeomorphic to the Klein bottle.

Problem 4. Let X be a finite connected simple graph, *i.e.*, a connected 1-dimensional CW complex with p 0-cells and q 1-cells with "no loops" and "no parallel edges". Compute $H_1(X)$ in terms of p and q. (You may use the fact that such a graph has a *spanning tree*: a subgraph which is a tree containing all the vertices of the original graph.)

Problem 5 (13–3, INVARIANCE OF DIMENSION). Prove that if $m \neq n$, then a nonempty topological space cannot be both an m-manifold and an n-manifold. (You may use Corollary 13.24 identifying the homology of $\mathbb{R}^n \smallsetminus \{0\}$; you may want to utilize [but still need to prove] the results from Problem 13–2.)