MATH 545: MANIFOLDS
HOMEWORK DUE FRIDAY WEEK 3

Problems taken from *Introduction to Topological Manifolds* are marked ITM x–y. Please review the syllabus for expectations and policies regarding homework.

Problem 1 (11–17). let $X \subseteq \mathbb{R}^3$ be the union of the unit 2-sphere with the line segment $\{(0,0,z) \mid -1 \leq z \leq 1\}$. Determine the universal covering space of X.

Problem 2 (11–20). Suppose X is a connected space that has a contractible universal covering space. For any connected and locally path-connected space Y, show that a continuous map $f\colon Y \to X$ is nullhomotopic if and only if for each $y \in Y$, the induced homomorphism $f_*\colon \pi_1(Y,y) \to \pi_1(X, f(y))$ is the trivial map. Give a counterexample to show that this result need not hold if the universal covering space is not contractible.

Problem 3 (11–21). For which compact, connected surfaces M do there exist continuous maps $f\colon M \to S^1$ that are not nullhomotopic? Prove your answer is correct. (You should use the result of Problem 11–20.)

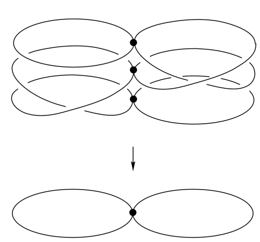
Problem 4 (12–2). Let $q\colon X_3 \to X_2$ be the covering map of Exercise 11.7 on p. 280. (You may assume that q is a covering map.)

(a) Determine the automorphism group $\mathrm{Aut}_q(X_3)$.
(b) Determine whether q is a normal covering.
(c) For each of the following maps $f\colon S^1 \to X_2$, determine whether f has a lift to X_3 taking 1 to 1:
  (i) $f(z) = z$,
  (ii) $f(z) = z^2$,
  (iii) $f(z) = 2 - z$,
  (iv) $f(z) = 2 - z^2$.

Problem 5 (12–3). Let X_n be the union of n circles described in Problem 10–9, and let A, B, C, and D denote the unit circles centered at 0, 2, 4, and 6, respectively. Define a covering map $q\colon X_4 \to X_2$ by

$$q(z) = \begin{cases} z & \text{if } z \in A, \\ 2 - (2-z)^2 & \text{if } z \in B, \\ (z-4)^2 & \text{if } z \in C, \\ z - 4 & \text{if } z \in D \end{cases}$$

as pictured. (You may assume that q is a covering map.)



(a) Identify the subgroup $q_*\pi_1(X_4, 1) \subseteq \pi_1(X_2, 1)$ in terms of the generators described in Example 11.17.
(b) Prove that q is not a normal covering map.