MATH 545: MANIFOLDS
FINAL EXAM PRACTICE PROBLEMS

Use these problems to prepare for your final oral exam. You are welcome to collaborate on them. I will ask you about at least one of these problems during your oral exam.

Problem 1. Let $X = S^1 \vee S^1$ and let $n > 2$. Give an explicit description of a covering map $p\colon Y \to X$ such that the image of $p_*\colon \pi_1(Y, y) \to \pi_1(X, x)$ is a free group on n generators and is a normal subgroup of index $n-1$. (You may describe Y by drawing a picture, but you must justify why p has the indicated properties.)

Problem 2. Recall (from the Math 544 final exam) that the *mapping torus* for a continuous map $f\colon X \to X$ is the quotient space

$$M_f := (X \times I)/{\sim}$$

where $\sim$ is the equivalence relation generated by $(x, 1) \sim (f(x), 0)$.

(a) Use the Mayer–Vietoris theorem to show that there is a long exact sequence of singular homology groups

$$\cdots \to H_n(X) \xrightarrow{1-f_*} H_n(X) \to H_n(M_f) \to H_{n-1}(X) \to \cdots .$$

(As part of your argument, you should identify the unlabeled arrows in the long exact sequence.)

(b) Compute the homology M_f where $f\colon S^2 \to S^2$ is the antipodal map.

Problem 3. Let M be a smooth manifold with chart (φ, U) around a point $p \in M$. Let $\mathcal{C}_p$ be the set of all smooth maps $\gamma\colon (-\varepsilon, \varepsilon) \to U$ such that $\gamma(0) = p$. Define an equivalence relation $\sim$ on $\mathcal{C}_p$ by declaring $\delta \sim \gamma$ if and only if

$$(\varphi \circ \delta)'(0) = (\varphi \circ \gamma)'(0).$$

Give $\mathcal{C}_p/{\sim}$ the structure of a real vector space and prove that T_pM and $\mathcal{C}_p/{\sim}$ are linearly isomorphic.

Problem 4. Let X denote the set of pairs of orthogonal lines through the origin in $\mathbb{R}^{n+1}$, viewed as a subspace of the product $\mathbb{RP}^n \times \mathbb{RP}^n$. Prove that X is an embedded submanifold of $\mathbb{RP}^n \times \mathbb{RP}^n$ and determine the dimension of X.

Problem 5. Let $f_1\colon M_1 \to N$ and $f_2\colon M_2 \to N$ be smooth maps. We say f_1 and f_2 *intersect transversely* when the product map $f_1 \times f_2\colon M_1 \times M_2 \to N \times N$ intersects the diagonal $\Delta_N = \{(q, q) \mid q \in N\} \subseteq N \times N$ transversely. Prove that if f_1 and f_2 intersect transversely, then the fiber product $F = (f_1 \times f_2)^{-1}\Delta_N$ is a submanifold of $M_1 \times M_2$, and for any $(p_1, p_2) \in F$,

$$T_{(p_1,p_2)}F = \{(v_1, v_2) \mid v_i \in T_{p_i}M_i,\ (df_1)_{p_1}(v_1) = (df_2)_{p_2}(v_2)\}.$$