MATH 544: TOPOLOGY
FRIDAY WEEK 10

Recall that S^1 is the unit circle centered at 0 in $\mathbb{R}^2$. Now consider some circles in space:

$$C_0 := S^1 \times 0,$$
$$C_1 := (3,0,0) + S^1 \times 0,$$
$$C_2 := (0,1/2,0) + 0 \times S^1.$$

Then we may form the following link[1] complement spaces

$$X := \mathbb{R}^3 \smallsetminus (C_0 \cup C_1),$$
$$Y := \mathbb{R}^3 \smallsetminus (C_0 \cup C_2).$$

More colloquially, X is the complement of two unlinked circles, and Y is the complement of two linked circles.

Problem 1. Apply the Seifert–van Kampen theorem to determine the fundamental groups of X and Y.

We are now going to physically model the spaces X and Y using the ambient universe and some carabiners. When a pair of carabiners is unlinked, the universe minus those carabiners is X; after linking the carabiners, we get Y.

Problem 2. Choose (path homotopy classes of) loops a, b such $\pi_1 X$ is generated by a and b.

(a) Use the provided cord and carabiners to model the loop $[a,b] = aba^{-1}b^{-1}$.
(b) What does your computation from Problem 1 tell you about $[a,b]$?

Problem 3. Retaining the cord configuration you created in Problem 2, carefully link the two carabiners so that you now have a loop in Y.

(a) Argue that the resulting loop in Y is $[c,d]$ for c,d generators of $\pi_1 Y$.
(b) What does your computation from Problem 1 tell you about $[c,d]$?
(c) Verify your assertion pulling on the cord.

Knot theory is the study of tame embeddings $S^1 \subseteq \mathbb{R}^3$ (or $S^1 \subseteq S^3$) up to ambient isotopy. Here *tame* means that the embedding can be extended to a solid torus (*i.e.* "thickened") and an *ambient isotopy* between knots $K, L\colon S^1 \hookrightarrow \mathbb{R}^3$ is a homotopy

$$H\colon \mathbb{R}^3 \times I \to \mathbb{R}^3$$

such that each $H(-,t)$ is a homeomorphism, $H(-,0) = \mathrm{id}_{\mathbb{R}^3}$, and $H(-,1) \circ K = L$. These definition can be extended to links (where S^1 is replaced by a disjoint union of circles).

The *knot group* of a link K is $\pi_1(\mathbb{R}^3 \smallsetminus K)$. In the above problems, we studied the knot groups of a trivial link with two components (*i.e.* $\pi_1 X$) and the knot group of the *Hopf link* (*i.e.* $\pi_1 Y$).

The *trefoil knot* T has parametrization

$$t \longmapsto ((2+\cos(3t))\cos(2t), (2+\cos(3t))\sin(2t), \sin(3t))$$

viewed as a path $[0,2\pi] \to \mathbb{R}^3$ (which naturally descends to an embedding $T\colon S^1 \cong \mathbb{R}/2\pi\mathbb{Z} \hookrightarrow \mathbb{R}^3$).

[1] A *link* is an embedding of a disjoint union of circles into space.

Problem 4. In this problem, you will determine the knot group of T.

(a) Show that T lives in a torus inside $\mathbb{R}^3$.

(b) Let U be an open thickening of the torus minus T and let V be the complement of an appropriate closed thickening of the torus (so that $U \cup V = \mathbb{R}^3 \smallsetminus T$). Use the Seifert–van Kampen theorem applied to U and V to prove that

$$\pi_1(\mathbb{R}^3 \smallsetminus T) \cong \langle a, b \mid a^2 = b^3 \rangle.$$

The group $\langle a, b \mid a^2 = b^3 \rangle$ is a presentation of the *braid group* B_3 on three strands. We will see it in a different guise next week.

Given a link diagram (a nice projection of your link onto the plane with under- and over-crossings recroded), the *Wirtinger presentation* allows you to produce a presentation for the associated knot group. The proof is a direct application of the Seifert–van Kampen theorem. See D. Rolfsen, *Knots and links*, §3D for details.