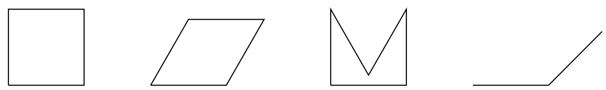
MATH 544: TOPOLOGY
WEDNESDAY WEEK 1

A (planar) *equilateral polygon* is a polygon (embedded in the plane) for which each side has the same length. You're holding an equilateral quadrilateral right now. We allow non-convexity and also degenerate configurations in which some vertices coincide. Below from left to right we see two equilateral quadrilterals, an equilateral pentagon, and a degenerate equilateral quadrilateral.



With your group, discuss and answer the following questions. Flag down the instructor if you have questions.

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(1) When should we consider two planar equilateral polygons to be "the same" geometrically?

(2) Let

$$M_n := \left\{ (p_1 = 1, p_2, \ldots, p_{n-1}, p_n = 0) \in \mathbb{C}^n \;\middle|\; \begin{array}{l} \text{for } z_k = p_{k+1} - p_k \\ |z_k| = 1 \text{ for } 1 \le k \le n-1 \end{array} \right\}.$$

In what sense should we consider M_n the "space" of planar equilateral n-gons up to similarity?

(3) Do we lose any information by rewriting M_4 as

$$M := \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 \;\middle|\; \begin{array}{l} |z_1| = |z_2| = |z_3| = 1, \\ 1 + z_1 + z_2 + z_3 = 0 \end{array} \right\}?$$

(4) Let $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle. By the first constraint, M_4 is a subset of $T^3 := S^1 \times S^1 \times S^1$, the 3-dimensional torus. Let $H = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid 1 + z_1 + z_2 + z_3 = 0\}$.
(a) What is the dimension of H?
(b) What is the expected dimension of $M = T^3 \cap H$?

(5) Fully describe the "shape" of M and draw a (cartoon) picture of it. (*Hint*: Play with your model and think about different (non-disjoint) pieces M might break into.)

It turns out that M_4 is *not* a manifold. One of our goals this term is to show that M_5 (the space of equilateral pentagons) is a compact 2-dimensional oriented manifold of genus 4.