Configuration spaces & braids

9.XII.22

For a space M,

$$F_n(M) := \{(m_1,\dots,m_n)\in M^n \mid m_i\neq m_j \text{ for } i\neq j\}$$

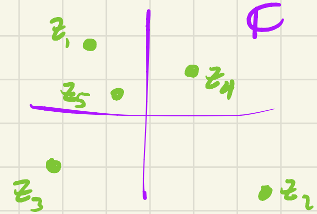
ordered configuration space

$$= M^n \setminus \Delta$$

"fat diagonal" $\{(m_1,\dots,m_n) \mid m_i = m_j \text{ for some } i\neq j\}$

topologized as a subspace of M^n (w/ product topology).

E.g. Here's a point in $M_5(\mathbb{C})$:



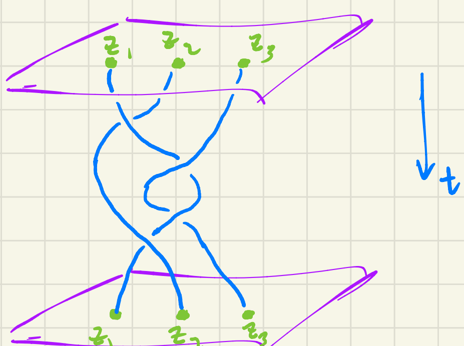
Action of $\mathfrak{S}_n$ on $F_n(M)$ permuting coords.

Quotient $F_n(M)/\mathfrak{S}_n =: C_n(M)$ is the unordered configuration space of n pts in M.

TPS Suppose M is a mfld of dimn d.
Why are $F_n(M)$, $C_n(M)$ manifolds? $\dim M^n = nd$
What are their dimensions? $= \dim F_n(M)$

Defn The *braid group* on n strands is $B_n := \pi_1 C_n(\mathbb{C})$
The *pure braid group* on n strands is $PB_n := \pi_1 F_n(\mathbb{C})$.

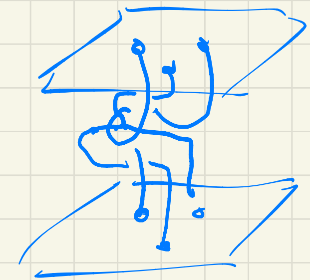
Here's a loop in $C_3(\mathbb{C})$:



In fact, this is also a loop in $F_3(\mathbb{C})$ since it doesn't permute the marked points.

Geometric braids Fix n & fix $z_1, \dots, z_n \in \mathbb{C}$ distinct.

Let $(f_1, \dots, f_n)$ be an n-tuple of cts maps $f_i : [0,1] \to \mathbb{C}$ s.t. $f_i(0) = z_i$, $f_i(1) = z_j$ for some $j = 1, \dots, n$, and s.t. the n **strands**



$$F_i : [0,1] \longrightarrow \mathbb{C} \times [0,1]$$
$$t \longmapsto (f_i(t), t)$$

have disjoint images.

The n strands are a **geometric braid**.

Call two braids F, G **isotopic** if there is an **ambient isotopy** pointwise fixing $\mathbb{C} \times \{0,1\}$ and taking F to G:

$$H : (\mathbb{C} \times [0,1]) \times [0,1] \longrightarrow \mathbb{C} \times [0,1]$$

- $H(-,t)$ homeo $\forall t$
- $H((-,0),t) = H((-,1),t) = id_{\mathbb{C}} \quad \forall t$
- $H(-,0) = id_{\mathbb{C}\times[0,1]}$
- $H(-,t)\circ F$ a braid $\forall t$ (no strand intersections)
- $H(-,1)\circ F = G$

$F: I\times\{1,\dots,n\} \longrightarrow \mathbb{C}\times[0,1]$

Each geometric braid induces a loop in $C_n(\mathbb{C})$ and isotopic geometric braids are homotopic as loops.

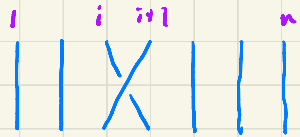
<u>Thm</u> (Artin) $\{\text{geometric } n\text{-strand braids}\}/\text{isotopy} \xrightarrow{\cong} B_n$

$\{\underbrace{\text{pure}}_{f_i(1)=z_i\ \forall i} \text{ geometric } n\text{-strand braids}\}/\text{isotopy} \xrightarrow{\cong} PB_n$

TPS What is the group operation on geometric braids?

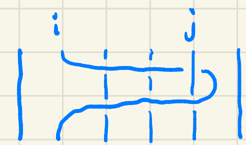
- multiply
- identity
- inverses?

Generators & relations



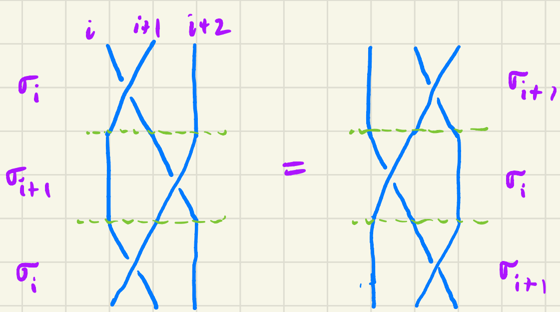
σ_i

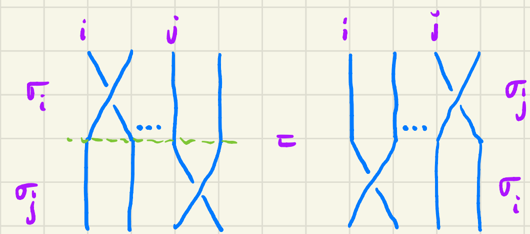
$i = 1, \dots, n-1$



$T_{i,j}$

$1 \leq i < j \leq n$





$|i-j| > 1$

Thm (Artin)

$$B_n \cong \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & 1 \le i \le n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i & |i-j| > 1 \end{array} \right\rangle$$

$$PB_n \cong \left\langle \begin{array}{c} T_{i,j} \\ 1 \le i < j \le n \end{array} \;\middle|\; \begin{array}{ll} [T_{p,q}, T_{r,s}] = 1 & p<q<r<s \\ [T_{p,s}, T_{q,r}] = 1 & p<q<r<s \\ T_{p,r} T_{q,r} T_{p,q} = T_{q,r} T_{p,q} T_{p,r} & \\ \qquad = T_{p,q} T_{p,r} T_{q,r} & p<q<r \\ [T_{r,s} T_{p,r} T_{r,s}^{-1}, T_{q,s}] = 1 & p<q<r<s \end{array} \right\rangle$$

Other incarnations Squarefree complex polynomials, Complements of hyperplane arrangements, Taffy pullers, mapping class groups, ETC!