7. $\overline{\text{XII}}$. 22

Function spaces

Note that $\mathrm{Set}(Z \times X, Y) \cong \mathrm{Set}(Z, \mathrm{Set}(X, Y))$.

$$\begin{tikzcd} Z \times X \arrow[d, "g"'] & & Z \arrow[d, "\hat{g}"] & z \arrow[d, mapsto] \\ Y \arrow[r, phantom, "\longmapsto"] & & \mathrm{Set}(X,Y) & g(z,-) : x \mapsto g(z,x) \end{tikzcd}$$

$$\begin{tikzcd} (z,x) \arrow[d, mapsto] & Z \times X \arrow[d, "\check{F}"'] & & Z \arrow[d, "F"] \\ (F(z))(x) & Y \arrow[r, phantom, "\longleftarrow\!\!\!\mid"] & & \mathrm{Set}(X,Y) \end{tikzcd}$$

Computer scientists like to call these "currying" & "uncurrying".

In algebra, a variant with $\otimes$ in place of $\times$ is the "Tensor-Hom adjunction":
$$\mathrm{Hom}(C \otimes A, B) \cong \mathrm{Hom}(C, \mathrm{Hom}(A,B))$$
(say for Hom = maps in Vect_k or Ab).

Q Is there a topology on $\mathrm{Top}(X,Y)$ such that

$$\forall Z \in \mathrm{Top}, \quad \mathrm{Top}(Z, \mathrm{Top}(X,Y)) \cong \mathrm{Top}(Z \times X, Y) \,?$$

{enrich? (homeo)}

└ bijection of sets

A Sometimes; if it exists, it's unique — called the exponential topology.

When $\mathcal{C}(X,Y) \in \mathcal{C}$ $\forall X,Y$ and $\mathcal{C}(Z, \mathcal{C}(X,Y)) \cong \mathcal{C}(\underbrace{Z \times X}_{\text{exists!}}, Y)$ $\forall Z$, then $\mathcal{C}$ is called Cartesian closed.

Top is not Cartesian closed. :(

In nice cases, the exponential topology is the "compact open topology".

Defn For X, Y spaces, $K \subseteq X$ compact, $U \subseteq Y$ open, define $S(K,U) := \{ f \in \mathrm{Top}(X,Y) \mid fK \subseteq U \}$. The sets $S(K,U)$ form the subbasis for a topology on $\mathrm{Top}(X,Y)$ called the compact open topology.

Note $\mathrm{Top}(X,Y) \subseteq Y^X = \prod_{x\in X} Y$. The subspace topology on $\mathrm{Top}(X,Y)$ inside Y^X is the "finite open topology" w/ subbasis $S(F,U)$, $F \subseteq X$ finite, $U \subseteq Y$ open.

Thm If X is loc compact H'ff and Y is any space, then the compact open topology on $\mathrm{Top}(X,Y)$ is exponential.

Pf **Step 1** If $g: Z\times X \longrightarrow Y$ is cts, then $\hat{g}: Z \longrightarrow \mathrm{Top}(X,Y)$ is cts: Need to show that for each $S(K,U)$,

$\hat{g}^{-1}S(K,U) = \{z\in Z \mid g(K,z) \subseteq U\} \subseteq Z$ is open. For $z \in \hat{g}^{-1}S(K,U)$, know $g^{-1}U \subseteq X\times Z$ open and contains $K\times\{z\}$. By the Tube Lemma (4.35) $\exists\, V, W$ open w/ $K\subseteq V$, $z\in W$ and $K\times\{z\} \subseteq V\times W \subseteq g^{-1}U$.

Now $z\in W \subseteq \hat{g}^{-1}S(K,U)$ as needed. ✓

Step 2 $\quad \mathrm{eval} : X \times \mathrm{Top}(X, Y) \longrightarrow Y$ is cts. $\quad$ {uses loc cpt H'ff hypothesis on X}

$$(x, g) \longmapsto g(x)$$

Step 3 Continuity of eval implies $F \longmapsto \check{F}$ takes cts F to cts $\check{F}$

(See Topology: A Categorical Approach 5.6.1 for details.) $\square$

E.g. For X loc cpt H'ff, a homotopies $H : X \times I \longrightarrow Y$ are in bijective corr with cts maps $\underbrace{\hat{H} : I \longrightarrow \underbrace{\mathrm{Top}(X, Y)}_{\text{compact open top}}}_{\text{"movie" of cts maps}}$.

E.g. For a pointed space (X, p), define $\Omega(X, p) := \mathrm{Top}_*((S^1, 1), (X, p))$ topologized as a subspace of $\mathcal{L}X := \mathrm{Top}(S^1, X)$ w/ cpt open top.

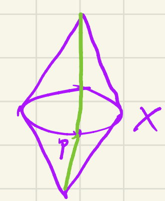
Q What is $\pi_0 \Omega(X,p)$?

A Suppose $f,g \in \Omega(X,p)$ and $\gamma : [0,1] \longrightarrow \Omega(X,p)$ is a path from f to g. Get cts $\gamma^\vee : [0,1] \times S^1 \longrightarrow X$ which is a path htpy from f to g. Similarly, path htpies between loops induce paths in $\Omega(X,p)$. Thus $\pi_0 \Omega(X,p) = \pi_1(X,p)$

This is a case of the suspension-loops adjunction:

$$\mathrm{Hot}_*(\Sigma(X,p),(Y,q)) \cong \mathrm{Hot}_*((X,p),\Omega(Y,q)).$$

where $\Sigma(X,p) = X \times I \big/ X\times 0 \cup X \times 1 \cup \{p\} \times I$.



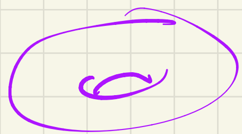
Fact $\Sigma S^n \simeq S^{n+1} \implies \pi_0 \underbrace{\Omega^n}_{\Omega \circ \Omega \circ \Omega \cdots \Omega}(X,p) \cong \pi_n(X,p)$.

n times

Midterm

5(c)

$$S^3 = D^2 \times S^1 \underset{S^1 \times S^1}{\cup} S^1 \times D^2$$



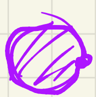
$(X \times Y)_k = \bigcup_{m+n=k} e^m \times e^n$

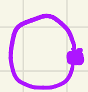
$e^m \in X_m$

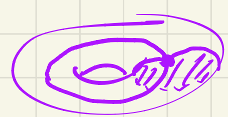
$e^n \in Y_n$

k-cells in

k	D^2	S^1	$D^2 \times S^1$	$S^1 \times D^2$	$S^1 \times S^1$
0	1	1	1	1	1
1	1	1	2	2	2
2	1	0	2	2	1
3	0	0	1	1	0







cell cpx X, Y, Z, cellular maps

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & X \\ Y & \end{tikzcd}$$

then $X \cup_Z Y$ has union cell structure

$$(X \cup_Z Y)_k = X_k \cup_{Z_k} Y_k$$

$$\begin{tikzcd} \cong \bar{B}^n \arrow[d, dash] & & \\ \Phi : D \arrow[rr] \arrow[dr] & & X \cup_Z Y \\ & X \arrow[r] & X \sqcup Y \arrow[u] \end{tikzcd}$$

k-cells in

k	D^2	S^1	$D^2 \times S^1$	$S^1 \times D^2$	$S^1 \times S^1$	S^3
0	1	1	1	1	1	1
1	1	1	2	2	2	2
2	1	0	2	2	1	3
3	0	0	1	1	0	2