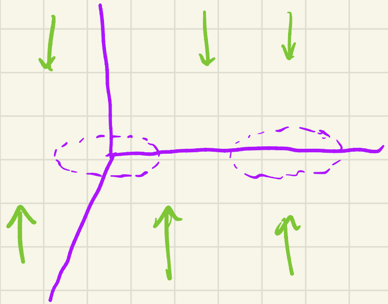
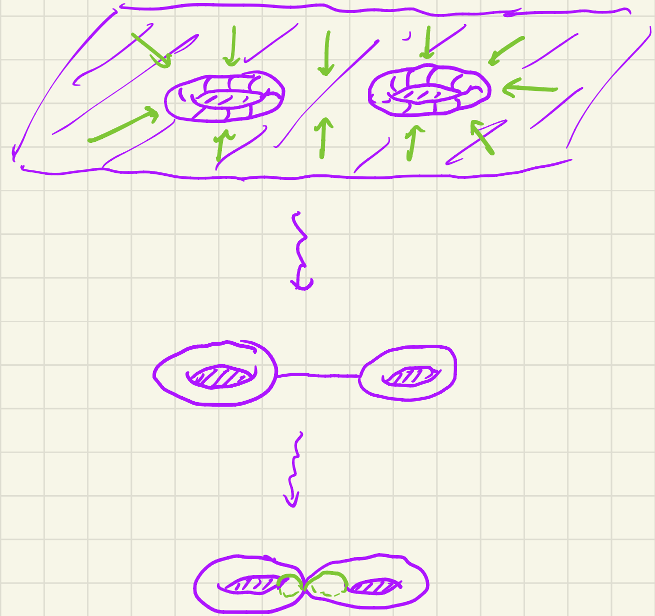
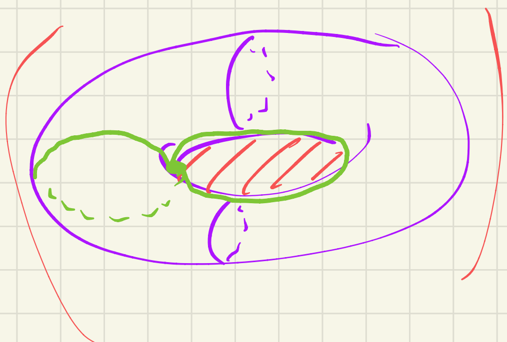
5. XII. 22





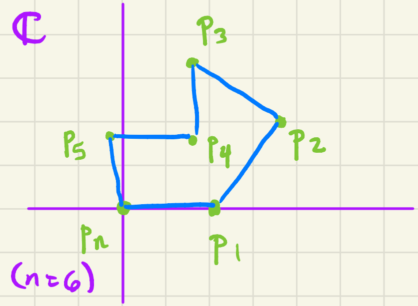
w/ $\pi_1 \cong \mathbb{Z} * \mathbb{Z}$

π_1  $= \mathbb{Z}$

Set $p_1 = 1$, $p_n = 0$.

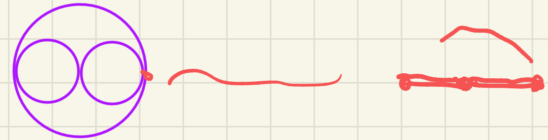
Recall $M_n := \{ (p_2, \dots, p_{n-1}) \in \mathbb{C}^{n-2} \mid |p_{k+1} - p_k| = 1 \text{ for } 1 \leq k \leq n-1 \}$
$\subseteq \mathbb{R}^{2n-4}$

Think of a point $p \in M_n$ as an equilateral n-gon:

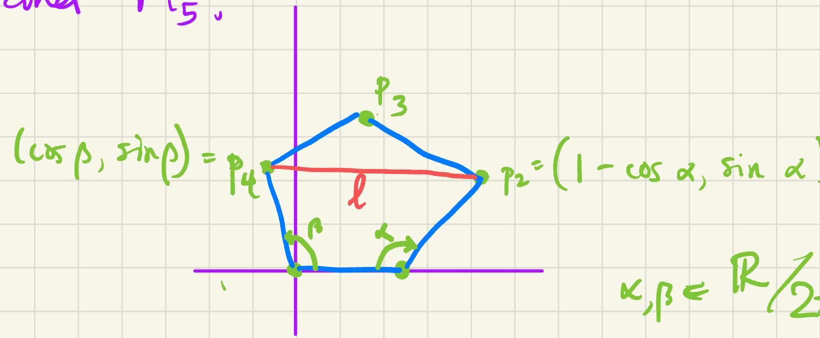


Give $M_n \subseteq \mathbb{C}^{n-2} \cong \mathbb{R}^{2n-4}$ the subspace topology.

Then M_n is the moduli space of unilateral n-gons (with vertices labelled).

From day 1: $M_4 \cong$ 

Goal Understand M_5.



$\alpha, \beta \in \mathbb{R}/2\pi\mathbb{Z} \cong S^1$

$0 < \ell < 2$: Exactly two possible values for p_3.

$\ell = 2$: p_3 is the midpoint of $\overline{p_2 p_4}$.

$\ell = 0$: p_3 is any point in $p_2 + S^1 = p_4 + S^1$ (occurs iff $\alpha = \beta = \pi/3$ or $5\pi/3$)

$2 < \ell$: $\nexists\, p_3$

Define $R \subseteq M_5$ to be pentagons with $0 < \ell \leq 2$

Define $D := \{ (\alpha, \beta) \in S^1 \times S^1 \mid 0 < \ell \leq 2 \}$ (Note $\ell^2 = (1 - \cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$.)

Then $\partial D = \{ (\alpha, \beta) \in S^1 \times S^1 \mid \ell = 2 \}$ and

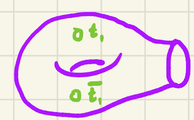
$$\begin{tikzcd}
\partial D \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & D \arrow[d] \\
D \arrow[r] & R
\end{tikzcd}$$

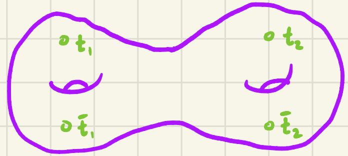
Set $\bar{D} = \{ (\alpha, \beta) \mid 0 \leq \ell \leq 2 \}$.

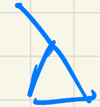
<u>Note</u> $\partial \bar{D} = \{ (\alpha, \beta) \mid \ell = 2 \}$
$\cong S^1$

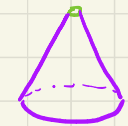
In fact, $\bar{D} \cong \mathbb{T}^2 \setminus \underbrace{e^2}_{\text{small open disk}}$

$$\Longrightarrow D \cong \mathbb{T}^2 \setminus \{ e^2 \cup \underbrace{t_1}_{\alpha = \beta = \frac{\pi}{3}} \cup \underbrace{\bar{t}_1}_{\alpha = \beta = \frac{5\pi}{3}} \}.$$

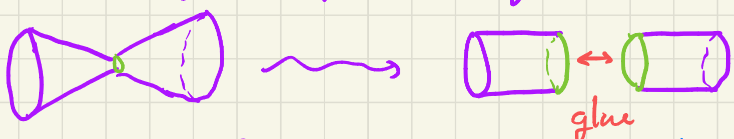


Thus $R \cong$ 

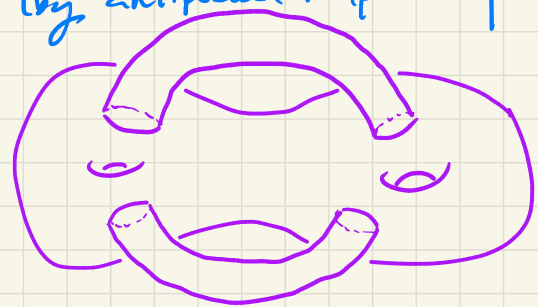


Model closed nbhds of t_1, t_2 by $CS^1 \setminus t_i$: 

In M_5, t_1, t_2 get replaced by circles glued together:



and similarly for $\bar{t}_1, \bar{t}_2$. (by antipodal map — preserves or'n)

Thus $M_5 \cong$ 

$\cong (\mathbb{T}^2)^{\# 4}$.

M. Freedman's
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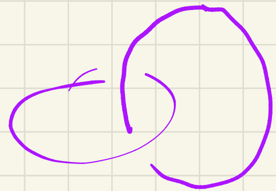
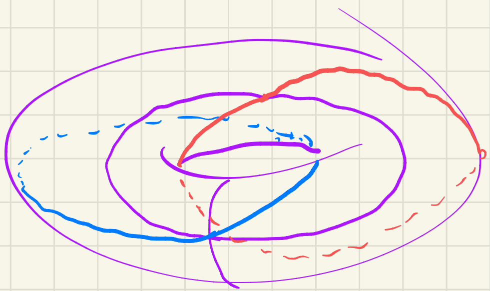
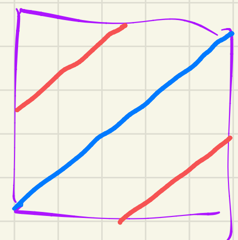
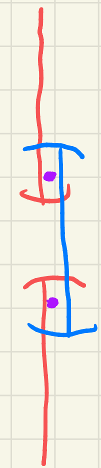
$\Big(\mathbb{R}^3 \setminus$  $\Big) = Y$

$\underline{Q}$ What is $\pi_1 Y$?







$U = (\text{Open thickening of } T^2) \setminus \text{Hopf link}$

$V = \mathbb{R}^3 \setminus (\text{smaller closed thickening of } T^2)$

$U \cap V$

U = shrink radius of torus & take exterior (unbdd) $\setminus HL$

V = dilate ~~r~~ radius of torus & take interior (bdd) $\setminus HL$

$U \cap V$ = fuzzy torus $\setminus$ Hopf link

$\simeq \mathbb{T}^2 \setminus$ Hopf link