Proof of SvK Space X, $U, V \subseteq X$ open, $U \cup V = X$, $U \cap V$ path connected.

For paths a, b, write $a \underset{S}{\sim} b$ for a, b paths in $S \subseteq X$ path homotopic in S.

Write $[a]_S$ for the class of a in $\pi_1(S, p)$. Note (base point always p)

$$
\begin{tikzcd}
U \cap V \arrow[r, hook, "j"] \arrow[d, "i"'] & V \arrow[d, "\ell"] \\
U \arrow[r, hook, "k"'] & X
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
{[a]_{U \cap V}} \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & {[a]_V} \arrow[ddd, mapsto] \\
& \pi_1(U \cap V) \arrow[r, "j_*"] \arrow[d, "i_*"'] & \pi_1(V) \arrow[d, "\ell_*"] & \\
& \pi_1(U) \arrow[r, "k_*"'] & \pi_1(X) & \\
{[a]_U} \arrow[rrr, mapsto] & & & {[a]_X}
\end{tikzcd}
$$

Notation: $\cdot$ = concatenation, $*$ = free product

e.g. $[a]_U * [b]_U * [c]_V = [a \cdot b]_U * [c]_V \in \pi_1 U * \pi_1 V$.

The unique map $\Phi : \pi_1 U * \pi_1 V \longrightarrow \pi_1 X$ is given by

$$\Phi([a_1]_U * [a_2]_V * \cdots * [a_{m-1}]_U * [a_m]_V)$$

$$= [a_1]_X \cdot [a_2]_X \cdots [a_{m-1}]_X \cdot [a_m]_X$$

$$= [a_1 \cdot a_2 \cdots a_{m-1} \cdot a_m]_X .$$

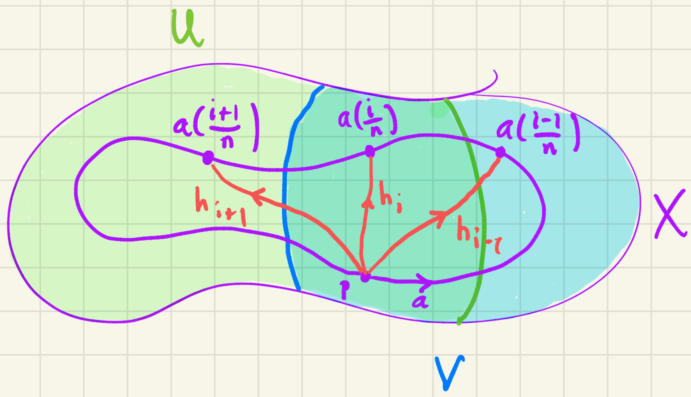
WTS: (1) Φ surj (2) $\bar{C} \subseteq \ker \Phi$ (3) $\ker \Phi \subseteq \bar{C}$

for $C = \{ (i_*\gamma)(j_*\gamma)^{-1} \mid \gamma \in \pi_1 U \cap V \}$

WLOG: replace U, V w/ their path components containing p.

(1) Φ surj: For $a : I \to X$ a loop at p, Lebesgue # lemma gives us $n \in \mathbb{Z}_+$ s.t. $a[\frac{i-1}{n}, \frac{i}{n}] \subseteq U$ or V. (Uses U, V open!) Let $a_i := a|_{[\frac{i-1}{n}, \frac{i}{n}]}$ reparam'd to have domain I. Then

$$[a]_X = [a_1 \cdot \cdots \cdot a_n]_X .$$



For $i=1,\dots,n-1$ choose h_i path $p \rightsquigarrow a(i/n)$ where if $a(i/n) \in \mathcal{U}\cap V$ then h_i lies in $\mathcal{U}\cap V$; o/w lies all in $\mathcal{U}$ or all in V.

(This uses $\mathcal{U}, V, \mathcal{U}\cap V$ path conn'd.) Set $\tilde{a}_i := h_{i-1} \cdot a_i \cdot \bar{h}_i$

(where $h_0, h_n = c_p$), so each $\tilde{a}_i$ is a loop at p lying entirely in $\mathcal{U}$ or V. Then $[a]_X = [\tilde{a}_1 \cdot \dots \cdot \tilde{a}_n]_X$, and

$\beta = [\tilde{a}_1]_{\mathcal{U} \text{ or } V} * [\tilde{a}_2]_{\mathcal{U} \text{ or } V} * \cdots * [\tilde{a}_n]_{\mathcal{U} \text{ or } V}$ ($\mathcal{U}$ or V according to where $\tilde{a}_i$ lies) satisfies $\Phi(\beta) = [a]_X$. ✓

② $\bar{C} \subseteq \ker \Phi$: Suffices to show $C \subseteq \ker \Phi$ b/c $\ker \Phi$ is a normal subgp.

Take $\gamma = [a]_{U\cap V} \in \pi_1(U\cap V)$. Then $\Phi\left((i_*\gamma) * (j_*\gamma)^{-1}\right) = \Phi([a]_U * [\bar{a}]_V)$
$= [a \cdot \bar{a}]_X = 1$.

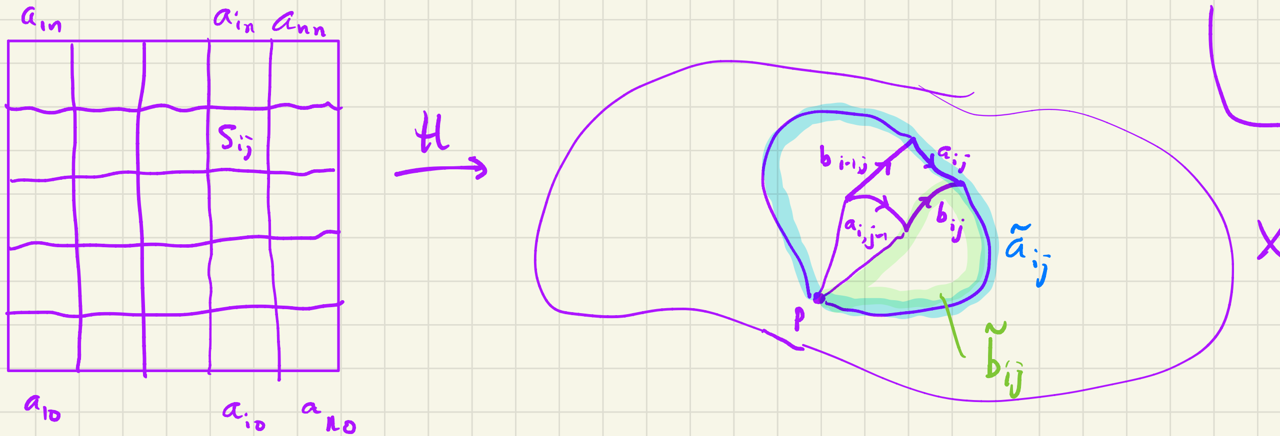
$$\begin{tikzcd} U\cap V \arrow[r, "j"] \arrow[d, "u"'] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd} \rightsquigarrow \begin{tikzcd} \gamma \in \pi_1(U\cap V) \arrow[r, "j_*"] \arrow[d, "i_*"'] & \pi_1 V \arrow[d] \\ \pi_1 U \arrow[r] & \pi_1 X \end{tikzcd}$$

③ $\ker \Phi \subseteq \bar{C}$ (the serious part!) : Let $\alpha = [a_1]_U * [a_2]_V * \cdots * [a_k]_V \in \pi_1 U * \pi_1 V$
and suppose $\Phi(\alpha) = e$. This means $a_1 \cdots a_k \underset{X}{\sim} c_p$. Need to show $\alpha \in \bar{C}$.

Let $H : I \times I \to X$ be a path htpy $a_1 \cdots a_k$ to c_p in X.

By Lebesgue # lemma, $\exists n \in \mathbb{Z}_+$ s.t. $S_{ij} := \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right]$ is mapped by H into U or V.

Let $v_{ij} = H\left(\frac{i}{n}, \frac{j}{n}\right)$ and let $a_{ij} = H\big|_{\left[\frac{i-1}{n}, \frac{i}{n}\right] \times \{j/n\}}$, $b_{ij} = H\big|_{\{i/n\} \times \left[\frac{j-1}{n}, \frac{j}{n}\right]}$
both reparam'd to have domain I.



$H \leq G$

$x \equiv e \ (H)$

$\Longleftrightarrow$

$xe^{-1} \in H$

$x \in H$

$H|_{I \times 0} = a_1 \cdots a_k$. By taking n to be a large power of 2, we can take endpts of the a_i to be of the form j/n

$\Rightarrow \quad H|_{I\times 0} \sim a_1 \cdots a_k \sim (a_{10} \cdots a_{q0}) \cdots (a_{r0} \cdots a_{n0})$.

$\Rightarrow \quad \alpha = [a_{10} \cdots a_{q0}]_U * \cdots * [a_{r0} \cdots a_{n0}]_V$

Take $h_{ij} : p \rightsquigarrow v_{ij}$ in $U \cap V$ if $v_{ij} \in U \cap V$, o/w in U or in V.

If $v_{ij} = p$, take $h_{ij} = c_p$. Define $\tilde{a}_{ij} = h_{i-1,j} \cdot a_{ij} \cdot \bar{h}_{ij}$

$\tilde{b}_{ij} = h_{0,j-1} \cdot b_{ij} \cdot \bar{h}_{ij}$ all in U or all in V.

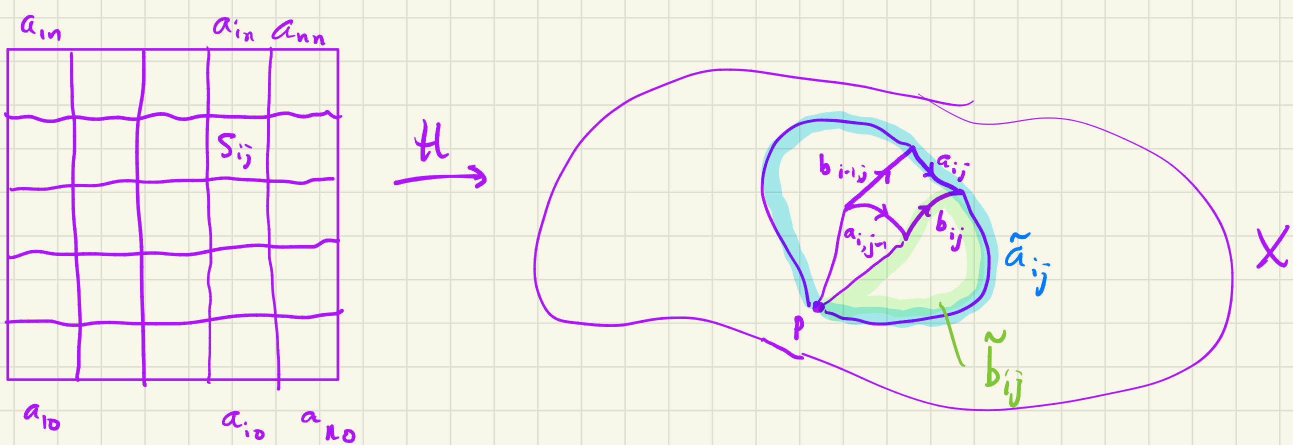
Then $\alpha = [\tilde{a}_{10}]_{U \text{ or } V} * [\tilde{a}_{20}]_{U \text{ or } V} * \cdots * [\tilde{a}_{n0}]_{U \text{ or } V}$

Strategy Work our way up the rows of the square showing $\alpha \equiv [\tilde{a}_{1j}]_{U \text{ or } V} * [\tilde{a}_{2j}]_{U \text{ or } V} * \cdots * [\tilde{a}_{nj}]_{U \text{ or } V} \mod \bar{C}$.

At the top we get c_p, whence $\alpha \in \bar{C}$ as desired.

Induction step: Assume $\alpha \equiv [\tilde{a}_{1,j-1}]_{U \text{ or } V} * \cdots * [\tilde{a}_{n,j-1}]_{U \text{ or } V} \mod \bar{C}$

Note that if a is a path in $U \cap V$, then $[a]_U \equiv [a]_V \mod \bar{C}$



a_{1n}
a_{in}
a_{nn}
S_{ij}
a_{10}
a_{i0}
a_{n0}
H
$b_{i-1,j}$
a_{ij}
$a_{i,j-1}$
b_{ij}
$\tilde{a}_{ij}$
p
$\tilde{b}_{ij}$
X

Suppose $H S_{ij} \subseteq V$. Then by the square lemma,

$$a_{i,j-1} \underset{V}{\sim} b_{i-1,j} \cdot a_{ij} \cdot \bar{b}_{ij}$$

$$\Longrightarrow \tilde{a}_{i,j-1} \underset{V}{\sim} \tilde{b}_{i-1,j} \cdot \tilde{a}_{ij} \cdot \bar{\tilde{b}}_{ij} \qquad \text{(h's cancel w/ $\bar{h}$'s)}$$

$$C = \{ [a]_U * [\bar{a}]_V \mid a \text{ loop in } U \cap V \}$$

For each factor $[\tilde{a}_{i,j-1}]_U$, check if S_{ij} is mapped into U or V. If V, then $\tilde{a}_{i,j-1}$ lands in $U \cap V$ so we can replace with $[\tilde{a}_{i,j-1}]_V$ mod $\bar{C}$. Correct each factor w/ S_{ij} mapping into U similarly.

Now each $[\tilde{a}_{i,j-1}]_V$ can be replaced by $[\tilde{b}_{i-1,j}]_V * [\tilde{a}_{ij}]_V * [\tilde{b}_{ij}]_V^{-1}$ and similarly for the factors in U. The $[\tilde{b}]$'s cancel and we get $\alpha \equiv [\tilde{a}_{ij}]_U * \cdots * [\tilde{a}_{nj}]_V$ mod $\bar{C}$ as desired. $\square$

$$\alpha \equiv [\tilde{a}_{1,j-1}]_u * [\tilde{a}_{2,j-1}]_v * \cdots * [\tilde{a}_{n,j-1}]_v$$

$$\equiv [\tilde{b}_{0,j}]_u * [\tilde{a}_{1,j}]_u * [\hat{b}_{1,j}]_u^{-1} * [\hat{b}_{1,j}]_v * \cdots$$

$$\equiv [\tilde{a}_{1,j}]_u * [\tilde{a}_{2,j}]_v * \cdots$$