α_i generate $\pi_1(\partial D_i, *_i)$, and let σ_i be an expression for $(\varphi_i)_*(\alpha_i) \in \pi_1(X_1, v)$ in terms of $\{\beta_i\}$. Then

$$\pi_1(X, v) \cong \langle \beta_1, \dots, \beta_n \mid \sigma_1, \dots, \sigma_k \rangle. \qquad \square$$

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π_1(compact surfaces)

Thm Let M be a space with polygonal presentation $\langle a_1, \dots, a_n \mid W \rangle$ with one face and one vx (after realization). Then

$$\pi_1(M) \cong \langle a_1, \dots, a_n \mid W \rangle.$$

Pf Under these hypotheses, M is a 2-dim'l CW cpx with 1-skeleton $M_1 \cong \bigvee_n S^1$. Thus $\pi_1(M_1) \cong \langle a_1, \dots, a_n \mid \varnothing \rangle$. The single 2-cell is attached via W, so $\pi_1(M) = \pi_1(M_2) \cong \langle a_1, \dots, a_n \mid W \rangle$. $\square$

Cor

- $\pi_1(S^2) \cong \langle \emptyset \mid \emptyset \rangle \cong e$
- $\pi_1((\mathbb{T}^2)^{\# n}) \cong \langle \beta_1, \gamma_1, \ldots, \beta_n, \gamma_n \mid \beta_1\gamma_1\beta_1^{-1}\gamma_1^{-1} \cdots \beta_n\gamma_n\beta_n^{-1}\gamma_n^{-1} = e \rangle$
- $\pi_1((\mathbb{RP}^2)^{\# n}) \cong \langle \beta_1, \ldots, \beta_n \mid \beta_1^2 \cdots \beta_n^2 = e \rangle$

Pf Standard presentations! $\square$

Note

- $\pi_1(\mathbb{T}^2) \cong \langle \beta, \gamma \mid \beta\gamma = \gamma\beta \rangle \cong \mathbb{Z} \times \mathbb{Z}$
- $\pi_1(\mathbb{RP}^2) \cong \langle \beta \mid \beta^2 = e \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

Goal Distinguish surfaces via their fundamental groups.

Tool Abelianization

Given a gp G, the commutator subgroup of G is

$$[G,G] := \langle \alpha\beta\alpha^{-1}\beta^{-1} \mid \alpha, \beta \in G \rangle \leq G.$$

Facts

- $[G,G] \trianglelefteq G$
- $[G,G] = e$ iff G Abelian
- $G^{ab} := G/[G,G]$ is Abelian (ITM writes $Ab(G)$)
- A hom $\varphi : G \to H$ induces $\varphi^{ab} : G^{ab} \to H^{ab}$ compatible with quotient maps.

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d] & H \arrow[d] \\ G^{ab} \arrow[r, dashed, "\varphi^{ab}"'] & H^{ab} \end{tikzcd}$$

Thm For G a group, H an Abelian gp, $\varphi : G \to H$ any hom, $\exists!$ hom $\tilde{\varphi} : G^{ab} \to H$ s.t.

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d] & H \\ G^{ab} \arrow[ur, dashed, "\tilde{\varphi}"'] & \end{tikzcd}$$

commutes. $\square$

$$\begin{tikzcd} S \arrow[r] \arrow[d] & G \\ F(S) \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Note $U : Ab \rightleftarrows Gp : (\ )^{ab}$ is an adjoint pair:

$$Ab(G^{ab}, H) \cong Gp(G, UH)$$
$$\tilde{\varphi} \longleftarrow \varphi$$

($U : Gp \rightleftarrows Set : F$ similar to

Prop
- $\pi_1(S^2)^{ab} \cong e$
- $\pi_1\left((T^2)^{\# n}\right)^{ab} \cong \mathbb{Z}^{2n}$
- $\pi_1\left((\mathbb{R}P^2)^{\# n}\right)^{ab} \cong \mathbb{Z}^{n-1} \times \mathbb{Z}/2\mathbb{Z}$.

Pf S^2: ✓

$(T^2)^{\# n}$: Let $G := \langle \beta_1, \gamma_1, \dots, \beta_n, \gamma_n \mid \prod_{i=1}^{n} [\beta_i, \gamma_i] = e \rangle$ be the standard presentation of $\pi_1((T^2)^{\# n})$ and define $\varphi: G^{ab} \longrightarrow \mathbb{Z}^{2n}$

$\beta_i \longmapsto e_i = (0, 0, \dots, 0, \underset{i\text{-th}}{1}, 0, \dots 0)$

$\gamma_i \longmapsto e_{i+n}$

(i.e. make this defn on $F(\beta_1, \gamma_1, \dots, \beta_n, \gamma_n)$ and note $\prod_{i=1}^{n} [\beta_i, \gamma_i] \longmapsto 0$ so get induced map φ on G^{ab})

$[x, y] = x y x^{-1} y^{-1}$

$\downarrow$

$f(x) f(y) f(x^{-1}) f(y^{-1})$

Since $\mathbb{Z}^{2n}$ is Abelian, get $\varphi: G^{ab} \longrightarrow \mathbb{Z}^{2n}$.

Now define $\psi : \mathbb{Z}^{2n} \to G^{ab}$

$$e_i \longmapsto \begin{cases} [\beta_i] & 1 \le i \le n, \\ [\gamma_{i-n}] & n+1 \le i \le 2n \end{cases}$$

equiv classes in G^{ab}

(This extends to a hom b/c $\mathbb{Z}^{2n}$ is free Abelian.)

Then φ, ψ are inverse homs.

$(\mathbb{RP}^2)^{\# n}$: Let $H := \langle \beta_1, \dots, \beta_n \mid \prod_{i=1}^{n} \beta_i^2 = e \rangle$ be the standard presentation of $\pi_1((\mathbb{RP}^2)^{\# n})$. Write $\langle f \rangle = \mathbb{Z}/2\mathbb{Z}$ (i.e. $f = 1 + 2\mathbb{Z} \in \mathbb{Z}/2\mathbb{Z}$).

Define $\varphi : H^{ab} \longrightarrow \mathbb{Z}^{n-1} \times \mathbb{Z}/2\mathbb{Z}$.

$$\beta_i \longmapsto e_i \qquad 1 \le i \le n-1$$

$$\beta_n \longmapsto f - e_{n-1} - \cdots - e_1$$

(Since $\varphi(\prod \beta_i^2) = 0$ this is well-defined on H and then descends to H^{ab}.)

$$\varphi\left(\prod \beta_i^2\right) = \sum_{i=1}^{n-1} 2e_i + 2f - 2e_{n-1} - \cdots - 2e_1$$

$$= 0$$

Define $\psi : \mathbb{Z}^{n-1} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow H^{ab}$. Then φ, ψ are inverse homs. $\square$

$$e_i \longmapsto [\beta_i]$$
$$f \longmapsto [\beta_1 \cdots \beta_n]$$

Thm (Classification of compact surfaces, Part II) Every nonempty compact conn'd 2-mfld is homeomorphic to exactly one of S^2, $(\mathbb{T}^2)^{\# n}$, or $(\mathbb{RP}^2)^{\# n}$.

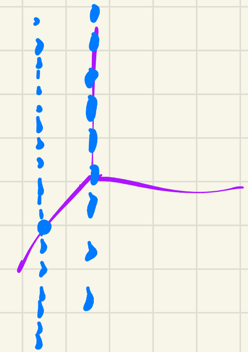
Pf Rank and torsion distinguish $\pi_1(\ )^{ab}$. $\square$

Cor Orientability and Euler characteristic are topological invariants of compact surfaces. $(\mathbb{RP}^2)^{\# n}$ is not orientable. $\square$

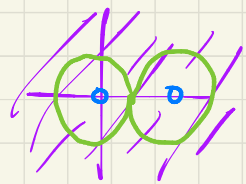
E.g. Consider $X = \mathbb{R}^3 \setminus \left(\{(0,0,z) \mid z \in \mathbb{R}\} \cup \{(1,0,z) \mid z \in \mathbb{R}\}\right)$

$Y = \mathbb{R}^3 \setminus (\text{linked circles})$

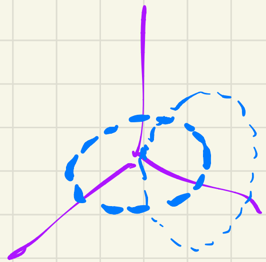
$= \mathbb{R}^3 \setminus \left(S^1 \times 0 \cup 0 \times ((1,0) + S^1)\right)$



$\simeq$



$\simeq S^1 \vee S^1$



$\simeq S^1 \times S^1$

$\uparrow$

Why?

Q1 What are the htpy types of X, Y?

Q2 $\pi_1(X), \pi_1(Y) = ?$

$\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$, $\pi_1(Y) \cong \mathbb{Z} \times \mathbb{Z}$