21.XI.22

Given $G_1 \xleftarrow{f_1} H \xrightarrow{f_2} G_2$ gp homs, let $G_1 \underset{H}{*} G_2 := (G_1 * G_2)/\bar{C}$ for $C = \{ f_1(g) f_2(g)^{-1} \mid g \in H \} \subseteq G_1 * G_2$ and $\bar{C}$ its normal closure; $G_1 \underset{H}{*} G_2$ is the amalgamated free product of $G_1, G_2$ along $H$.

Prop $G_1 \underset{H}{*} G_2$ is the pushout of $G_1 \xleftarrow{f_1} H \xrightarrow{f_2} G_2$ in Gp, i.e.

$$\begin{tikzcd}
H \arrow[r, "f_2"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & G_2 \arrow[d, "\iota_2"] \arrow[ddr, bend left, "\varphi_2"] & \\
G_1 \arrow[r, "\iota_1"'] \arrow[drr, bend right, "\varphi_1"'] & G_1 \underset{H}{*} G_2 \arrow[dr, dashed, "\exists! \varphi"] & \\
& & K
\end{tikzcd}$$

Pf Given $\varphi_1, \varphi_2$ making the diagram commute, we get a hom $\psi : G_1 * G_2 \to K$ s.t. $\psi(g_i) = \varphi_i(g_i)$ since $G_1 * G_2$ is coproduct in Gp. If $g \in H$, $\iota_1 f_1(g) = \iota_2 f_2(g)$ i.e. $f_1(g) f_2(g)^{-1} = e \in G_1 * G_2$. Thus

$C \subseteq \ker \psi \trianglelefteq G_1 \underset{H}{*} G_2 \implies \bar{C} \subseteq \ker \psi$. Thus $\psi$ induces

$$\begin{tikzcd} G_1 * G_2 \arrow[r, "\psi"] \arrow[d] & K \\ G_1 \underset{H}{*} G_2 \arrow[ur, "\varphi"'] & \end{tikzcd}$$

by univ property of quotients, and the $\varphi \iota_i = \varphi_i$ by def'ns. Uniqueness follows from $G_1 \sqcup G_2$ generating $G_1 \underset{H}{*} G_2$. $\square$

<u>Thm</u> For $G_1 \xleftarrow{f_1} H \xrightarrow{f_2} G_2$ gp homs and presentations

$$G_1 \cong \langle \alpha_1, \dots, \alpha_m \mid \rho_1, \dots, \rho_r \rangle$$
$$G_2 \cong \langle \beta_1, \dots, \beta_n \mid \sigma_1, \dots, \sigma_s \rangle$$
$$H \cong \langle \gamma_1, \dots, \gamma_p \mid \tau_1, \dots, \tau_t \rangle,$$

(Note: Just need $\gamma_1, \dots, \gamma_p$ generate.)

we have $G_1 \underset{H}{*} G_2 \cong \langle \alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \mid \rho_1, \dots, \rho_r, \sigma_1, \dots, \sigma_s, u_1 = v_1, \dots, u_p = v_p \rangle$

for $u_\alpha = f_1(\gamma_\alpha)$ in terms of $\{\alpha_i\}$, $v_\alpha = f_2(\gamma_\alpha)$ in terms of $\{\beta_i\}$. $\square$

In particular, if $H = e$, then $G_1 \underset{H}{*} G_2 \cong G_1 * G_2$.

## Thm (Seifert-van Kampen)

$X = U \cup V$ a space, $U, V \subseteq X$ open, $U \cap V$ path conn'd, $p \in U \cap V$ then the inclusion maps

$U \hookleftarrow U \cap V \hookrightarrow V$ induce an isomorphism

$$\pi_1(X,p) \cong \pi_1(U,p) \underset{\pi_1(U\cap V,p)}{*} \pi_1(V,p).$$

George Seifert
1897 - 1996

Egbert van Kampen
1908 - 1942

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In diagrams, 

$$\begin{tikzcd} U\cap V \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

induces

$$\begin{tikzcd} \pi_1(U\cap V,p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \pi_1(V,p) \arrow[d] \arrow[ddr, bend left] & \\ \pi_1(U,p) \arrow[r] \arrow[drr, bend right] & \pi_1(U,p) \underset{\pi_1(U\cap V,p)}{*} \pi_1(V,p) \arrow[dr, dashed, "\cong"] & \\ & & \pi_1(X,p) \end{tikzcd}$$

or, more simply,

$$\begin{tikzcd} \pi_1(U \cap V, p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \pi_1(V, p) \arrow[d] \\ \pi_1(U, p) \arrow[r] & \pi_1(X, p) \end{tikzcd}.$$

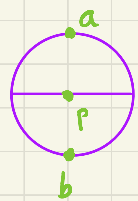
<u>Idea</u> Decompose loops in $X$ into concatenations / words of loops in $U$ and loops in $V$ to get elements of $\pi_1(U,p) * \pi_1(V,p)$ then remember that loops in $U \cap V$ have two different names!


Proof deferred — applications first!

**Special case 1** If $U \cap V$ is simply conn'd, then $\pi_1(X,p) \cong \pi_1(U,p) * \pi_1(V,p)$.

**Special case 2** If $U$ is simply conn'd, then $\pi_1(X,p) \cong \pi_1(V,p)/\overline{H}$

for $H = j_* \pi_1(U \cap V, p)$. (Here $j : U \cap V \hookrightarrow V$.)

**E.g.** $X =$ 


$U = X \setminus a$, $V = X \setminus b$ are both $\simeq S^1$

$U \cap V \simeq *$

$$\pi_1(X,p) \cong \pi_1(U,p) * \pi_1(V,p) \cong \mathbb{Z} * \mathbb{Z}.$$

**Wedge Sums** $p \in X$ is a nondegenerate base point if $p$ has a nbhd admitting a strong deformation retract onto $p$ ($U \xrightarrow{r} p$ s.t. $ir \simeq id_U$ rel $p$).

**Lemma** Suppose $p_i \in X_i$ is a nondegenerate base point for $i = 1, \dots, n$. Then $* := [p_i]$ is a nondegenerate basepoint of $X_1 \vee \cdots \vee X_n$.

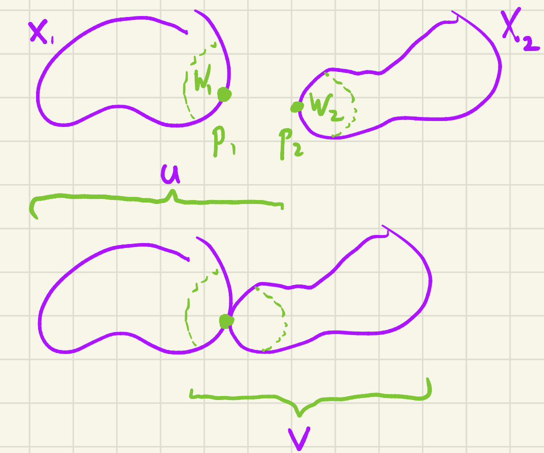
**Pf** For each $i$, choose nbhd $W_i$ of $p_i$ in $X_i$ admitting a strong def'n retraction $r_i : W_i \to p_i$ and let $H_i : W_i \times I \to W_i$ be the associated htpy $id_{W_i} \simeq \iota_{p_i} r_i$. Define $H : (\coprod W_i) \times I \to \coprod W_i$ with $H|_{W_i \times I} = H_i$.

Now $q : \coprod X_i \to \bigvee X_i$ restricted to $\coprod W_i$ is a quotient onto a nbhd $W$ of $*$ $\Rightarrow$ $q \times id_I$ is a quotient map and $q \circ H$ respects the identifications so descends to $(\bigvee W_i) \times I$ yielding a strong def retract onto $*$. $\square$

**Thm** Let $X_1, \dots, X_n$ be spaces with nondegenerate base points $p_j \in X_j$. Then the map $\mathop{*}_{i=1}^{n} \pi_1(X_i, p_i) \to \pi_1(\bigvee X_i, *)$ induced by $\iota_j : X_j \hookrightarrow \bigvee X_i$ ($j = 1, \dots, n$) is an isomorphism.

Pf $n=2$ Choose nbhds $W_i$ in which $p_i$ is a strong def retract, and let

$U = q(X_1 \amalg W_2)$, $V = q(W_1 \amalg X_2)$.




Then $U, V \subseteq X_1 \vee X_2$ are open and

$$* \longrightarrow U \cap V$$
$$X_1 \hookrightarrow U$$
$$X_2 \hookrightarrow V$$

are htpy equivs.

Since $U \cap V$ is simply conn'd,

special case 1 $\Rightarrow$

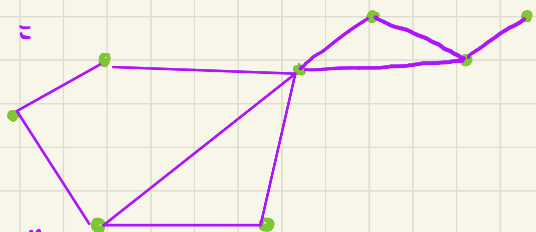
$$\begin{array}{ccc} \pi_1(U,*) * \pi_1(V,*) & \xrightarrow{\cong} & \pi_1(X_1 \vee X_2, *) \\ \cong \uparrow & & \\ \pi_1(X_1,p_1) * \pi_1(X_2,p_2) & & \end{array}$$

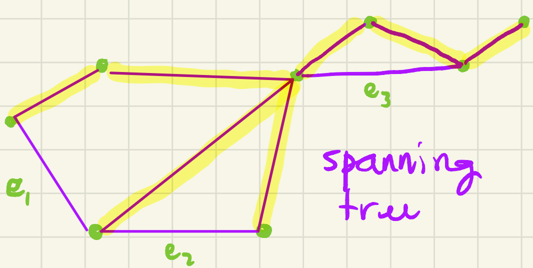
For the genl case, induce. $\square$

E.g. $\pi_1(\underbrace{S^1 \vee \cdots \vee S^1}_{n}, *) \cong F([\omega_1], [\omega_2], \dots, [\omega_n])$.

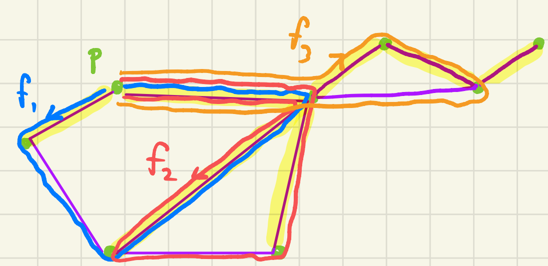
**Graphs** (informally)

A graph is a CW complex of dimn $\leq 1$ (undirected, not necessarily simple).

$G =$ 




$\rightsquigarrow$

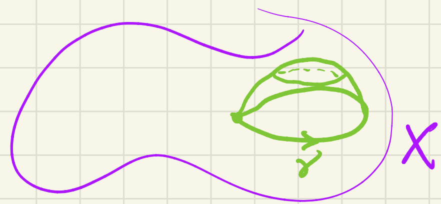


By SvK, $\pi_1(G, p) \cong F([f_1], [f_2], [f_3])$.

This works for all finite conn'd graphs: $\pi_1(G, p) \cong$ free gp on $n$ generators for $n = \#$ edges not in spanning tree of conn'd component of $p$.

## $\pi_1$( CW cpxs )

**Prop** (Attaching a disk) $X$ = path conn'd space.




$$
\begin{tikzcd}
\partial D \arrow[r, "\varphi"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\
D \arrow[r] & \tilde{X}
\end{tikzcd}
$$

closed 2-cell $\simeq D$,

$v \in \partial D$, $\tilde{v} = \varphi(v) \in X$, $\gamma = \varphi_*(\alpha) \in \pi_1(X, \tilde{v})$ for $\langle \alpha \rangle = \pi_1(\partial D, v)$.

Then $\pi_1(X, \tilde{v}) \overset{\text{surj}}{\twoheadrightarrow} \pi_1(\tilde{X}, \tilde{v})$ induced by $X \hookrightarrow \tilde{X}$ with kernel $\overline{\langle \gamma \rangle}$.

If $\pi_1(X,\tilde{v}) \cong \langle \beta_1,\dots,\beta_n \mid \sigma_1,\dots,\sigma_s \rangle$ then

$$\pi_1(\tilde{X},\tilde{v}) \cong \langle \beta_1,\dots,\beta_n \mid \sigma_1,\dots,\sigma_s,\tau \rangle$$

for $\tau$ an expression for $\gamma$ in terms of $\{\beta_i\}$.

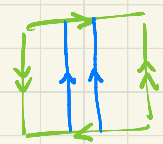
$\{D$ kills $\gamma$

E.g.

$$\begin{tikzcd} S^1 \arrow[r, "\omega^2", "\tau"'] \arrow[d] & S^1 \arrow[d] \\ D^2 \arrow[r] & \mathbb{RP}^2 \end{tikzcd}$$

so $\pi_1(\mathbb{RP}^2) = \omega^{\mathbb{Z}}/\omega^{2\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z}$ } torsion!!

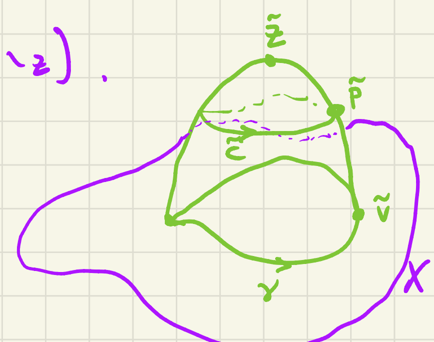
Exc Check geometrically that



is path nullhomotopic.

Pf Sketch Take $z \in D^\circ$, $U = D^\circ$, $V = X \sqcup (D \setminus z)$.

Apply SvK to $\underset{\text{"dome"}}{\tilde{U}} \simeq *$, $\underset{\tilde{X}\setminus\tilde{z}}{\tilde{V}} = q(V) \subseteq \tilde{X}$.




Then $\pi_1(\tilde{X},\tilde{p}) \cong \pi_1(\tilde{V},\tilde{p}) / \overline{\langle [\tilde{c}] \rangle}$

by special case 2. Use a path from $\tilde{p}$ to $\tilde{v}$ to get the statement in terms of basepoint $\tilde{v}$ and quotient by $\overline{\langle \gamma \rangle}$.

Finally $X$ is a def retract of $\tilde{V}$ (b/c $\partial D$ is a def retract of $D \setminus z$) so we get the theorem. $\square$

**Prop** Attaching an $n$-cell along its boundary does not alter $\pi_1$.

**Pf** Same but $\tilde{U} \cap \tilde{V} \cong B^n \setminus 0$ is simply conn'd. $\square$

**Thm** $X$ a conn'd finite CW cpx, $v \in X_1$ is contained in the closure of every 2-cell. Let $\beta_1, \dots, \beta_n$ generate $\pi_1(X_1, v) \cong F(\beta_1, \dots, \beta_n)$ and let $e_1, \dots, e_k$ be the 2-cells of $X$. For each $i = 1, \dots, k$ let $\Phi_i : D_i \to X$ be a characteristic map for $e_i$ taking $v_i \in \partial D_i$ to $v$. Let $\varphi_i = \Phi_i|_{\partial D_i} : \partial D_i \to X$ be the attaching map, let

$\alpha_i$ generate $\pi_1(\partial D_i, *_i)$, and let $\sigma_i$ be an expression for $(\varphi_i)_*(\alpha_i) \in \pi_1(X_1, v)$ in terms of $\{\beta_i\}$. Then

$$\pi_1(X, v) \cong \langle \beta_1, \ldots, \beta_n \mid \sigma_1, \ldots, \sigma_k \rangle. \qquad \square$$