18.XI.22

Free products (informally)

write e_α for identity of G_α.

Indexed family of groups $(G_\alpha) = (G_\alpha)_{\alpha \in A}$.

disjoint union

A word of length $m \geq 0$ in (G_α) is an element of $\left(\coprod_{\alpha \in A} G_\alpha\right)^m$ written $(g_1, \dots, g_m)$. The length 0 word is $()$. Let $W = \coprod_{m \geq 0} \left(\coprod_{\alpha \in A} G_\alpha\right)^m$ be the set of words.

Define a product on W via $(g_1, \dots, g_m)(h_1, \dots, h_n) := (g_1, \dots, g_m, h_1, \dots, h_n)$.

Note So far no group structure invoked!

An elementary reduction is an operation of the form

$$(g_1, \dots, g_i, g_{i+1}, \dots, g_m) \longmapsto (g_1, \dots, \underbrace{g_i g_{i+1}}_{\text{mult in } G_\alpha}, \dots, g_m) \quad \text{if } g_i, g_{i+1} \in \text{same } G_\alpha$$

$$(g_1, \dots, g_i, \underbrace{e_\alpha}_{\text{id of } G_\alpha}, g_{i+2}, \dots, g_m) \longmapsto (g_1, \dots, g_i, g_{i+2}, \dots, g_m).$$

Declare two words to be elementarily equivalent if one is an elementary reduction of the other, and let $\sim$ denote the equivalence rel'n generated by elementary equivalence.

Defn The free product of (G_α) is $\underset{\alpha\in A}{*} G_\alpha := W/\sim$ with mult'n induced by mult'n of words.

Q What is the identity elt of $*G_\alpha$?

Notation Write $G_1 * \cdots * G_n$ for $A = \{1, \dots, n\}$. E.g. $G*H$ is the free product of G, H.

Call a word reduced if it cannot be shortened by elementary reduction.

Thm Every element of $*G_\alpha$ is represented by a unique reduced word.

Pf Read 9.2. $\square$

E.g. $A = \varnothing$, free product is e.

$A = $ singleton, $\underset{\{1\}}{*} G = G$

$C_2 * C_2$ consists of alternating strings of β, γ;

$(\beta\gamma\beta\gamma\beta\gamma)(\gamma\beta\gamma\beta) = \beta\gamma$.

generator β, $\beta^2 = e$ — generator γ, $\gamma^2 = e$

Thm $* = $ coproduct in Grp when equipped with $\iota_\beta : G_\beta \longrightarrow \underset{\alpha \in A}{*} G_\alpha$ for each $\beta \in A$.

$g \longmapsto g$

In particular,

$$\begin{tikzcd} & H \\ & \underset{\alpha\in A}{*} G_\alpha \arrow[u, dashed, "\exists! f"'] \\ G_\beta \arrow[uur, "f_\beta"] \arrow[ur, "\iota_\beta"'] & \end{tikzcd}$$

$SL_2\mathbb{Z}/\{\pm I\} = PSL_2\mathbb{Z} \cong C_2 * C_3$

i.e. if $(f_\alpha : G_\alpha \to H)_{\alpha \in A}$ is a family of homs w/ common target H

then $\exists! f : * G_\alpha \longrightarrow H$ s.t. $f \iota_\alpha = f_\alpha$ $\forall \alpha \in A$. As such, the free product is the unique group satisfying this universal property.

Pf Define $f(g_1) = f_\alpha(g_1)$ if $g_1 \in G_\alpha$,

$$f(g_1 \cdots g_m) = f(g_1) \cdots f(g_m).$$

Check that this is the unique hom making the diagrams commute. $\square$

Free groups

Given an object σ, define $F(\sigma) := \sigma^{\mathbb{Z}}$, the infinite cyclic group generated by σ. Given a set S, the free group on S is

$$F(S) := \mathop{\ast}_{\sigma \in S} F(\sigma).$$

Elements of $F(S)$ can be represented by reduced words $\sigma_1^{n_1} \cdots \sigma_k^{n_k}$ where $\sigma_i \neq \sigma_{i+1} \in S$. & $n_i \in \mathbb{Z} \setminus \{0\}$

E.g. $F(\emptyset) = e$, $F(\sigma) \cong \mathbb{Z}$, $F(\sigma, \tau) \cong \mathbb{Z} * \mathbb{Z}$

Define $\iota : S \hookrightarrow F(S)$, $\sigma \mapsto \sigma^1 = \sigma$

Thm (Universal property of free groups) For any set S, group G, and function $f : S \to G$, $\exists!$ hom $\tilde{f} : F(S) \to G$ s.t. $\tilde{f}(\sigma) = f(\sigma)$ $\forall \sigma \in S$, i.e.

$$\begin{tikzcd} S \arrow[d, hook, "\iota"'] \arrow[dr, "f"] & \\ F(S) \arrow[r, dashed, "\exists! \tilde{f}"'] & G \end{tikzcd}.$$

⚠ ι, f are just functions
$\tilde{f}$ is a hom.

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d] & T \arrow[d] \\ FS \arrow[r, dashed] & FT \end{tikzcd}$$

Pf Define $\tilde{f}(\sigma_1^{n_1} \cdots \sigma_k^{n_k}) = f(\sigma_1)^{n_1} \cdots f(\sigma_k)^{n_k}$ $\forall \sigma_1, \dots, \sigma_k \in S$, $n_1, \dots, n_k \in \mathbb{Z}$.
Check that $\tilde{f}$ works and it's the only hom that does. $\square$

$$\boxed{\begin{array}{l} F : \mathbf{Set} \rightleftarrows \mathbf{Top} : U \\ \quad S \longmapsto (S, \text{discrete}) \end{array}}$$

Note This is an example of an adjunction:

$$F : \mathbf{Set} \underset{\perp}{\rightleftarrows} \mathbf{Gp} : U$$

free — forgetful/underlying

$$\begin{tikzcd} S \arrow[d] & \longmapsto & FS \arrow[d] \\ T & & FT \end{tikzcd}$$

$$\mathbf{Set}(S, UG) \cong \mathbf{Gp}(FS, G).$$

$$f \longmapsto \tilde{f}$$

"natural" in S and G

Presentations of groups For a set S and $R \subseteq F(S)$, define

$$\underbrace{\langle \underset{\text{generators}}{S} \mid \underset{\text{relations}}{R} \rangle}_{\text{group presentation}} := F(S) / \overline{R} .$$

$\overline{R}$ normal closure of R: smallest normal subgp containing R

Note If $S \subseteq G$ generates G, then we get a surjective hom

$$f: F(S) \twoheadrightarrow G \quad \text{so} \quad G \cong F(S)/\ker(f). \quad \text{If } R \subseteq F(S) \text{ with}$$

$$\begin{matrix} s & \longmapsto & s \\ \cap & & \\ S & & \end{matrix}$$

$\overline{R} = \ker(f)$, then $\underbrace{G \cong \langle S \mid R \rangle}_{\text{presentation of } G}$.

If G admits a presentation w/ S finite, call G **finitely presented**.

E.g.

- $F(S) \cong \langle S \mid \emptyset \rangle$
- $\mathbb{Z} \times \mathbb{Z} \cong \langle \beta, \gamma \mid \underbrace{\beta\gamma = \gamma\beta}_{\text{shorthand for } \beta\gamma\beta^{-1}\gamma^{-1}} \rangle$
- $\mathbb{Z}/n\mathbb{Z} \cong C_n = \langle \alpha \mid \alpha^n = 1 \rangle$
- $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \langle \beta, \gamma \mid \beta^2 = \gamma^3 = 1, \beta\gamma = \gamma\beta \rangle$
- $PSL_2\mathbb{Z} \cong \langle \beta, \gamma \mid \beta^2 = \gamma^3 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$

  (2×2 integer matrices w/ det 1 modulo $\{\pm I_2\}$)

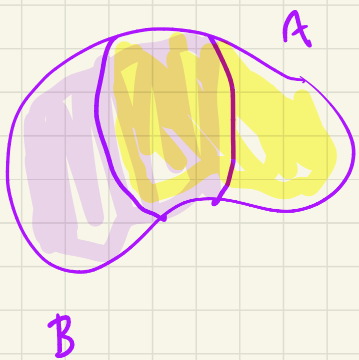
Moral Exc Check

first 4 presentations or read 9.13.

Note $\underbrace{\text{Undecidable problems}}_{\nexists \text{ algorithm answering these questions}}$: Is $\langle S \mid R \rangle \cong \langle S' \mid R' \rangle$? (isomorphism problem)

In $\langle S \mid R \rangle$, is a given word represent e? (word problem)

(Skipping free Abelian groups. Read pp. 244-248. Most follows from $\oplus$ = coproduct in Ab.)



$$\begin{tikzcd} A\cap B \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

$$\rightsquigarrow \pi_1$$

$$\begin{tikzcd} \pi_1 A\cap B \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \pi_1 A \arrow[d] \arrow[ddr, bend left] & \\ \pi_1 B \arrow[r] \arrow[drr, bend right] & P \arrow[dr, dashed] & \\ & & \pi_1 X \end{tikzcd}$$

For $A\cap B$ conn'd, Seifer-van Kampen tells us how to compute $\pi_1 X$ in terms of $\pi_1 A$, $\pi_1 B$, $\pi_1 A\cap B$

$\exists!$ hom $P \to \pi_1 X$

SvK: it's an iso

<u>Cor</u> $\pi_1(X\vee Y)$

π_1(surfaces)

$\vdots$