other lift, then $\varepsilon(\tilde{f}(s)) = \varepsilon(\tilde{f}'(s)) \Rightarrow \tilde{f}(s) - \tilde{f}'(s) \in \mathbb{Z} \ \forall s$.

Since I is connected, $\tilde{f} - \tilde{f}'$ cts, know $\tilde{f} - \tilde{f}'$ is constant. $\square$

16.XI.22

Cor 2 (Path lifting criterion for S^1) Suppose $f_0, f_1 : I \to S^1$ with same endpoints, and $\tilde{f}_0, \tilde{f}_1$ are lifts w/ same initial point. Then $f_0 \sim f_1$ iff $\tilde{f}_0(1) = \tilde{f}_1(1)$.

I.e. $f_0 \sim f_1$ iff they have the same net angular change!

Pf If $\tilde{f}_0, \tilde{f}_1$ have the same terminal point then they are path htpic since $\mathbb{R}$ is simply conn'd. Thus $f_0 = \varepsilon \tilde{f}_0$, $f_1 = \varepsilon \tilde{f}_1$ are path htpic.

Now suppose $H : f_0 \sim f_1$. By htpy lifting,

$$\begin{tikzcd} I \times 0 \arrow[r, "\tilde{f}_0"] \arrow[d] & \mathbb{R} \arrow[d, "\varepsilon"] \\ I \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & S^1 \end{tikzcd}$$

and $\tilde{H} : \tilde{f}_0 \sim \underbrace{\tilde{H}(-,1)}_{\text{lift of } f_1 \text{ starting at } \tilde{f}_0(0)}$.

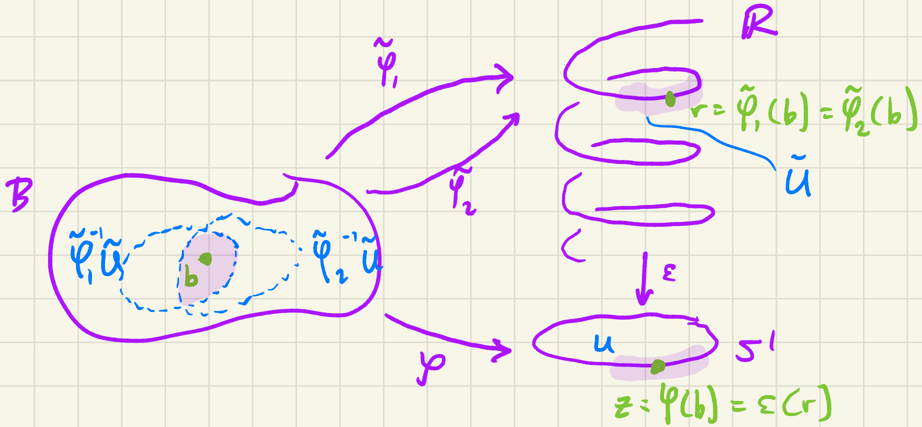
By uniqueness of lifts, $\tilde{H}(-,1) = \tilde{f}_1$ so $\tilde{f}_0 \sim \tilde{f}_1$. $\square$

Proofs of Key Thms:

I. Unique lifting for S^1: Suppose B conn'd, $\varphi: B \to S^1$, $\tilde{\varphi}_1, \tilde{\varphi}_2 : B \to \mathbb{R}$ lifts of φ agreeing at some point of B. Then $\tilde{\varphi}_1 = \tilde{\varphi}_2$.

Pf Set $A = \{b \in B \mid \tilde{\varphi}_1(b) = \tilde{\varphi}_2(b)\}$. Know $A \neq \emptyset$ by hypothesis. Since B is conn'd, suffices to show A is open and closed.

A open: Suppose $b \in A$ and write $r := \tilde{\varphi}_1(b) = \tilde{\varphi}_2(b)$, $z := \varepsilon(r) = \varphi(b)$.



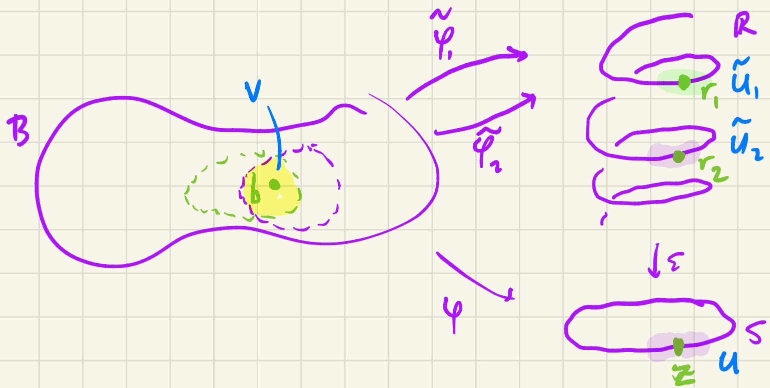
Take $U \subseteq S^1$ evenly covered open nbhd of z, $\tilde{U}$ component of $\varepsilon^{-1}U$ containing r.

On $V := \tilde{\varphi}_1^{-1}\tilde{U} \cap \tilde{\varphi}_2^{-1}\tilde{U}$, $\tilde{\varphi}_1$ & $\tilde{\varphi}_2$ take values in $\tilde{U}$. Since ε inj on $\tilde{U}$, $V \subseteq A$

$\Longrightarrow$ A open.

$\mathcal{A}$ closed: Show $B \setminus \mathcal{A}$ open. Take $b \in B \setminus \mathcal{A}$, $r_1 = \tilde{\varphi}_1(b)$, $r_2 = \tilde{\varphi}_1(b)$

so $r_1 \neq r_2$. Set $z = \varepsilon(r_1) = \varepsilon(r_2) = \varphi(b)$. Take $U \subseteq S^1$ evenly covered open



nbhd of z, $\tilde{U}_1$, $\tilde{U}_2$ components of $\varepsilon^{-1}U$ containing r_1, r_2, resp.

Set $V = \tilde{\varphi}_1^{-1}\tilde{U}_1 \cap \tilde{\varphi}_2^{-1}\tilde{U}_2$ and observe $\tilde{\varphi}_1 V \subseteq \tilde{U}_1$, $\tilde{\varphi}_2 V \subseteq \tilde{U}_2$, $\tilde{U}_1 \cap \tilde{U}_2 = \emptyset$ so $V \subseteq B \setminus \mathcal{A}$.

Thus $B \setminus \mathcal{A}$ open, $\mathcal{A}$ closed. $\square$

Now go on to prove II.

II. Homotopy lifting property for S^1: Suppose B is a locally conn'd space, $\varphi_0, \varphi_1 : B \to S^1$, $H : \varphi_0 \simeq \varphi_1$, $\tilde{\varphi}_0$ a lift of φ_0. Then $\exists!\ \tilde{H}$ s.t.

$$\begin{tikzcd}
B \times 0 \arrow[r, "\tilde{\varphi}_0"] \arrow[d] & \mathbb{R} \arrow[d, "\varepsilon"] \\
B \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & S^1 .
\end{tikzcd}$$

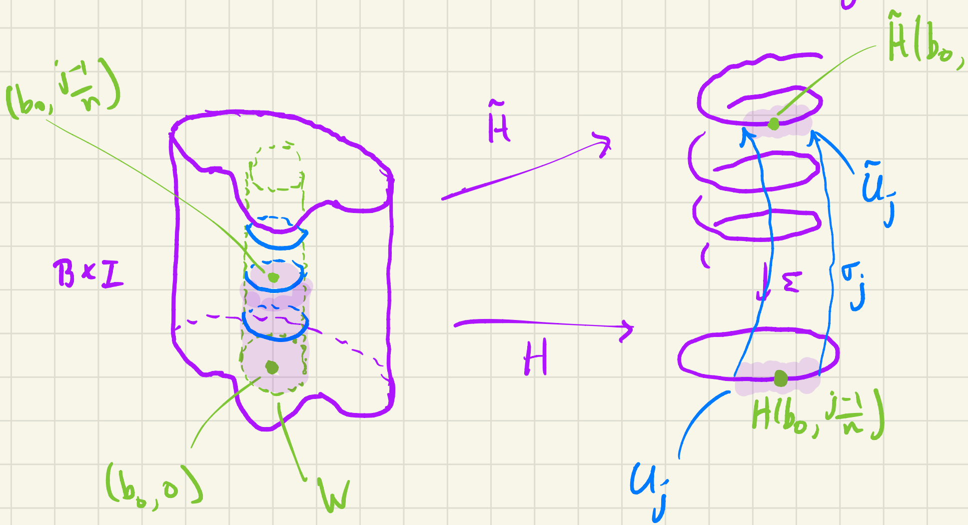
If H is stationary on some $A \subseteq B$, then so is $\tilde{H}$.

Pf Uniqueness follows from I: If $\tilde{H}, \tilde{H}'$ are two lifts, then for each $b \in B$, $\tilde{H}(b,-)$, $\tilde{H}'(b,-)$ are lifts of $H(b,-)$ $\overset{I}{\Rightarrow}$ $\tilde{H}, \tilde{H}'$ agree on $\{b\} \times I$ $\Rightarrow \tilde{H} = \tilde{H}'$. The same argument works for $\tilde{H}, \tilde{H}'$ only defined on $W \times I$ for any $W \subseteq B$.

Existence: Fix $b_0 \in B$. For each $s \in I$, take U an evenly covered nbhd of $H(b_0, s)$. There exist open $V \subseteq B$, $J \subseteq I$ such that

$(b_0, s) \in V \times J \subseteq H^{-1}U$. The collection of all such $V \times J$ (s varying) is an open cover of $\{b_0\} \times I$, ← compact, so we can take a finite subcover $V_1 \times J_1, \dots, V_m \times J_m$. Let W be a connected open nbhd of b_0 (exists by local conn'dness!) contained in $V_1 \cap \cdots \cap V_m$, and let δ be a Lebesgue number of the open cover $J_1, \dots, J_m$ of I.

Take $n \in \mathbb{Z}_+$ s.t. $\frac{1}{n} < \delta$. Then for $j = 1, \dots, n$, $H(W \times [\frac{j-1}{n}, \frac{j}{n}]) \subseteq U$. (evenly covered)



Choose an evenly covered open $U_1 \subseteq S^1$ containing $H(W \times [0, \frac{1}{n}])$ and take $\sigma_1 : U_1 \to \mathbb{R}$ a local section of ε over U_1 s.t. $\sigma_1(\varphi_0(b_0)) = \tilde{\varphi}_0(b_0)$. For $(b, s) \in W \times [0, 1/n]$, define $\tilde{H}(b,s) = \sigma_1(H(b,s))$. This lifts

H on $W \times [0, \frac{1}{n}]$ and $\tilde{H}(b,0) = \tilde{\varphi}_0(b)$ for $b \in W$ by I.

Suppose now for induction that $\tilde{H}$ has been defined on $W \times [0, \frac{j-1}{n}]$.

Let U_j be an evenly covered open in S^1 containing $H(W \times [\frac{j-1}{n}, \frac{j}{n}])$ and take $\sigma_j : U_j \to \mathbb{R}$ local section of ε over U_j s.t. $\sigma_j(H(b_0, \frac{j-1}{n})) = \tilde{H}(b_0, \frac{j-1}{n})$.

Define $\tilde{H}(b,s) := \sigma_j(H(b,s))$ for $(b,s) \in W \times [\frac{j-1}{n}, \frac{j}{n}]$.

This agrees on the overlap $W \times \{\frac{j-1}{n}\}$ by I (check!) and the glueing lemma gives us a lift to $W \times [0, \frac{j}{n}]$. By induction, we get a lift $\tilde{H}$ defined on $W \times I$. Use I + glueing to extend to $B \times I$. By construction, $\tilde{H}(b,0) = \tilde{\varphi}_0(b)$.

Finally, if H is stationary on $A \subseteq B$, then $\forall a \in A$, $H(a,-) = c_{\varphi_0(a)}$ w/ unique lift starting at $\tilde{\varphi}_0(a)$ equal to $c_{\tilde{\varphi}_0(a)}$. Thus $\tilde{H}$ is also stationary on A. $\square$

Defn Given a loop $f: I \to S^1$, choose a lift $\tilde{f}: I \to \mathbb{R}$ of f. Then

$N(f) := \tilde{f}(1) - \tilde{f}(0) \in \mathbb{Z}$ is the winding number of f.

Notes $\rightarrow$

- in $\mathbb{Z}$ b/c $\varepsilon(\tilde{f}(0)) = f(0) = f(1) = \varepsilon(\tilde{f}(1))$
- Lifts differ by a constant integer so winding number is well-defined.

E.g. $N(\text{const}) = 0$, $N(\varepsilon|_I) = 1$, $N(t \mapsto \varepsilon(2t)) = 2$.

Thm Loops $f, g: I \to S^1$ both based at p are path htpic iff $N(f) = N(g)$.

Pf By the path lifting property, f, g have lifts $\tilde{f}, \tilde{g}: I \to \mathbb{R}$ with $\tilde{f}(0) = \tilde{g}(0)$. By the path lifting criterion, $f \simeq g$ iff $\tilde{f}(1) = \tilde{g}(1)$. This is equiv to $N(f) = N(g)$. $\square$

Write $\omega := \varepsilon|_I : t \mapsto \exp(2\pi i t)$.

Thm $\pi_1(S^1, 1)$ is an infinite cyclic group generated by $[\omega]$.

In particular, $\pi_1(S^1, p) \cong \mathbb{Z}$ $\forall_p \in S^1$.

Pf Define

$$\begin{array}{rcl} J : \mathbb{Z} & \rightleftarrows & \pi_1(S^1, 1) : K \\ n & \longmapsto & [\omega]^n \\ N(f) & \longleftarrow & [f] \end{array}$$

well-defined by previous thm

homomorphism:
$[\omega]^{m+n} = [\omega]^m [\omega]^n$

Suffices to prove K is a 2-sided inverse to J.

For $n \in \mathbb{Z}$, define $\alpha_n : I \to S^1$, $t \mapsto \exp(2\pi i n t)$ so that $[\alpha_n] = [\omega]^n$

and $N(\alpha_n) = n$.

Thus $K(J(n)) = K([\alpha_n]) = N(\alpha_n) = n$ and

$$J(K([f])) = J(N(f)) = [\omega]^{N(f)} = [\alpha_n] = [f].$$

$n = N(f)$ same winding # $\square$

<u>Cor</u> $\pi_1(\mathbb{C}^*, 1) \cong \mathbb{Z}$ and two loops in $\mathbb{C}^*$ are path htpic

$[f] \longmapsto N\left(\frac{f}{|f|}\right)$ iff they have the same basepoint and same winding number. $\square$

<u>Cor</u> $\pi_1(\mathbb{T}^n, (1, \ldots, 1)) \cong \mathbb{Z}^n$. $\square$

<u>Cor</u> For $n \geq 2$, $S^n \not\cong \mathbb{T}^n$.

<u>Pf</u> $\pi_1(S^n) = e$ but $\pi_1(\mathbb{T}^n)$ is nontrivial. $\square$

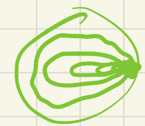
<u>Reading</u>: Degree Thy for S^1, pp. 227-229.

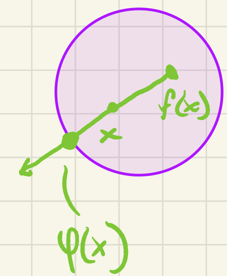
Brouwer Fixed Point Thm Every cts map $f: \bar{\mathbb{B}}^2 \to \bar{\mathbb{B}}^2$ has a fixed point ($x \in \bar{\mathbb{B}}^2$ s.t. $f(x) = x$).

Pf Suppose f has no fixed point and define $\varphi: \bar{\mathbb{B}}^2 \to S^1$

(well-defined since $f(x) \neq x \ \forall x$).

Then $\varphi|_{S^1} = \mathrm{id}_{S^1}$.





$$\begin{tikzcd} I \arrow[r] \arrow[d, bend right] & X \\ 0,1 \arrow[r, mapsto] & p \\ S^1 = I/_{0\sim 1} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

$[\mathrm{id}_{S^1}] \in \pi_1(S^1, 1)$

$\updownarrow$

$[\omega] \neq 0 \in \mathbb{Z} \cong \pi_1(S^1)$

$$\begin{tikzcd} I \arrow[r] & S^1 \\ S^1 \arrow[r] & S^1 \end{tikzcd} \qquad \updownarrow$$

$$\begin{tikzcd} I \arrow[r] \arrow[d, "\varepsilon"'] & S^1 \\ S^1 \arrow[ur, dashed] & \end{tikzcd}$$

$\omega: t \mapsto \exp(2\pi i t)$

But this shows $[\mathrm{id}_{S^1}]$ extends to $\bar{\mathbb{B}}^2$ so it's nullhomotopic ☠. $\square$

Pf <u>Redux</u> Construct $\varphi : \bar{B}^2 \longrightarrow S^1$ as above with $\varphi|_{S^1} = id_{S^1}$.

Then
$$\begin{tikzcd} S^1 \arrow[r, hook, "\iota"] \arrow[dr, "id_{S^1}"'] & \bar{B}^2 \arrow[d, "\varphi"] \\ & S^1 \end{tikzcd}$$
commutes so
$$\begin{tikzcd} \pi_1(S^1, 1) \arrow[r, "\iota_*"] \arrow[dr, "id_* = id"'] & \pi_1(\bar{B}^2, 1) \arrow[d, "\varphi_*"] \\ & \pi_1(S^1, 1) \end{tikzcd}$$
commutes.

But $\bar{B}^2$ is simply conn'd, so $\pi_1(\bar{B}^2, 1) = e$

$\Longrightarrow \varphi_* \iota_*$ is the trivial map

This is a ⨳ as $id : \pi_1(S^1, 1) \circlearrowleft$ is not trivial. $\square$