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Loose end:

Prop $f \in \mathrm{Top}(X,Y)$ is a htpy equiv iff $[f] \in \mathrm{Hot}(X,Y)$ is an isomorphism.

Pf If $f : X \simeq Y : g$, then $gf \simeq id_X \Rightarrow [g][f] = [gf] = [id_X]$

and $fg \simeq id_Y \Rightarrow [f][g] = [fg] = [id_Y]$,

so $[f] : X \cong Y \in \mathrm{Hot}$.

Now check that $[g] = [f]^{-1} \in \mathrm{Hot}(Y,X)$ means that g is a htpy inverse to f. ✓ $\square$

Products Given an indexed family $(X_\alpha)_{\alpha \in A}$ of objects in $\mathcal{C}$, an object $P = \prod_{\alpha \in A} X_\alpha$ equipped with $\pi_\alpha \in \mathcal{C}(P, X_\alpha)$ for $\alpha \in A$ is the product of (X_α) when $\forall W \in \mathrm{Ob}\,\mathcal{C}$, $f_\alpha \in \mathcal{C}(W, X_\alpha)$ for $\alpha \in A$, $\exists! f \in \mathcal{C}(W,P)$ s.t.

$$\begin{tikzcd} & P \arrow[d, "\pi_\alpha"] \\ W \arrow[ur, dashed, "f"] \arrow[r, "f_\alpha"'] & X_\alpha \end{tikzcd}$$

commutes $\forall \alpha \in A$.

In the binary case ($|A|=2$) this looks like

$$\begin{tikzcd} & W \arrow[dl, bend right, "f_1"'] \arrow[d, dashed, "\exists! f"] \arrow[dr, bend left, "f_2"] & \\ X_1 & P \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & X_2 \end{tikzcd}$$

and we write $P = X_1 \times X_2$.

E.g.

- In **Set**, Cartesian product = categorical product
- In **Top**, Cartesian product w/ product topology
- In **Gp**, Cartesian product w/ coordinatewise mult'n.

Thm If $(\prod X_\alpha, (\pi_\alpha))$ exists in $\mathcal{C}$, then it is unique up to unique iso respecting the projection morphisms. I.e., if $(P, (p_\alpha))$ is also the product of (X_α), then $\exists!$ iso $f : \prod X_\alpha \to P$ s.t. $\pi_\alpha = p_\alpha f$ $\forall \alpha \in A$.

Pf

$$\begin{tikzcd} & \prod X_\alpha \arrow[d, "\pi_\alpha"] \arrow[dl, dashed, bend left, "\exists! f"] \\ P \arrow[ur, dashed, bend left, "\exists! g"] \arrow[r, "p_\alpha"'] & X_\alpha \end{tikzcd}$$

Further,

$$\begin{tikzcd} & \prod X_\alpha \arrow[d] \\ \prod X_\alpha \arrow[ur, shift left, "gf"] \arrow[ur, shift right, "id"'] \arrow[r] & X_\alpha \end{tikzcd}$$

and

$$\begin{tikzcd} & P \arrow[d] \\ P \arrow[ur, shift left, "fg"] \arrow[ur, shift right, "id"'] \arrow[r] & X_\alpha \end{tikzcd}$$

so both composites are identities. $\square$

$$
\begin{tikzcd}
& \prod X_\alpha \arrow[dl, "\exists! f"'] \arrow[d, "\pi_\alpha"] \arrow[dd, bend right=100, "fg"'] \\
P \arrow[r, "p_\alpha"'] \arrow[dr, "\exists! g"'] & X_\alpha \\
& \prod X_\alpha \arrow[u, "\pi_\alpha"']
\end{tikzcd}
$$

Coproducts Given an indexed family $(X_\alpha)_{\alpha \in A}$ of objects in $\mathcal{C}$, an object $S = \coprod_{\alpha \in A} X_\alpha$ equipped with morphisms $\iota_\alpha : X_\alpha \to S$ for $\alpha \in A$ s.t. $\forall W \in \mathrm{Ob}\,\mathcal{C}$, $f_\alpha : X_\alpha \to W$ for $\alpha \in A$, $\exists! f : S \to W$ such that

$$\begin{tikzcd} X_\alpha \arrow[d, "\iota_\alpha"'] \arrow[dr, "f_\alpha"] & \\ S \arrow[r, dashed, "\exists! f"'] & W \end{tikzcd}$$

commutes.

In the binary case:

$$\begin{tikzcd} X_1 \arrow[dr, "\iota_1"] \arrow[ddr, bend right, "f_1"'] & & X_2 \arrow[dl, "\iota_2"'] \arrow[ddl, bend left, "f_2"] \\ & X_1 \amalg X_2 \arrow[d, dashed, "\exists! f"] & \\ & W & \end{tikzcd}$$

Thm Coproducts are unique up to unique iso respecting the inclusion maps. $\square$

E.g.

- In Set, $\sqcup$ = disjoint union.
- In Top, $\sqcup$ = disjoint union (w/ disjoint union topology).
- In Top_*, $\sqcup = \vee$, wedge sum. (See HW.)
- In Ab, $\sqcup = \oplus$, the direct sum. (See HW.)
- In Gp, $\sqcup$ = free product. (Upcoming.)

Pullbacks and pushouts

Given
$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$
in $\mathcal{C}$, the pullback $X \times_{f,g} Y = X \times_Z Y$ is
$$\begin{tikzcd} X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] \arrow[dr, phantom, "\lrcorner", very near end] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$
such that
$$\begin{tikzcd} W \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] \arrow[dr, phantom, "\lrcorner", very near end] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$
.

E.g. • In Set, $X \times_Z Y = \{(x,y) \in X \times Y \mid f(x) = g(y)\}$.

• Same in Top. For $f : U \hookrightarrow X \hookleftarrow V : g$,

$$U \times_X V = U \cap V \qquad \text{(check topology matches!)}$$

Pushouts are dual:

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \arrow[ddr, bend left] & \\
X \arrow[r] \arrow[drr, bend right] & X \cup_Z Y \arrow[dr, dashed, "\exists!"] & \\
& & W
\end{tikzcd}$$

E.g. In Set or Top,

$$X \cup_Z Y = X \sqcup Y \big/ f(z) \sim g(z) .$$

Circular reasoning

Goal $\pi_1(S^1, 1) \cong \mathbb{Z}$

Idea Measure angular change of a loop φ to a path in $\mathbb{R}$ along $\varepsilon : \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$:

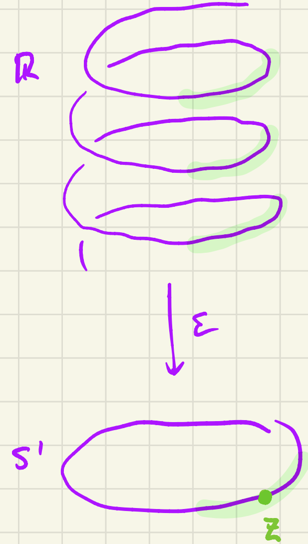
$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\varepsilon"] \\ I \arrow[r, "\varphi"'] \arrow[ur, dotted, "\tilde{\varphi}"] & S^1 \end{tikzcd}$$

Then $\Theta(x) = 2\pi \tilde{\varphi}(x)$ measures angular change.

Lifts do not always exist.

Prop Each point $z \in S^1$ has a nbhd U which is evenly covered by ε :
$\varepsilon^{-1} U$ is a countable union of disjoint open intervals $\tilde{U}_n$ s.t. $\varepsilon|_{\tilde{U}_n} : \tilde{U}_n \cong U$.

Sketch

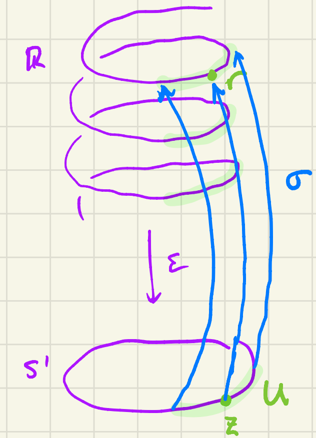


Cor (Local sections for S^1) If $U \subseteq S^1$ is evenly covered open, then $\forall z \in U$ and $r \in \varepsilon^{-1}\{z\}$, $\exists\, \sigma$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\varepsilon"] \\ U \arrow[r, hook] \arrow[ur, dashed, "\sigma"] & S^1 \end{tikzcd}$$

s.t. $\sigma(z) = r$.

Pf Take $\tilde{U} = \tilde{U}_n \ni r$ and σ to be the inverse of $\varepsilon : \tilde{U} \cong U$. $\square$



Key Thms

I. Unique lifting for S^1: Suppose B conn'd, $\varphi : B \to S^1$, $\tilde{\varphi}_1, \tilde{\varphi}_2 : B \to \mathbb{R}$ lifts of φ agreeing at some point of B. Then $\tilde{\varphi}_1 = \tilde{\varphi}_2$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\varepsilon"] \\ B \arrow[ur, shift left, "\tilde{\varphi}_1"] \arrow[ur, shift right, "\tilde{\varphi}_2"'] \arrow[r, "\varphi"'] & S^1 \end{tikzcd}$$

II. Homotopy lifting property for S^1: Suppose B is a locally conn'd space, $\varphi_0, \varphi_1 : B \to S^1$, $H : \varphi_0 \simeq \varphi_1$, $\tilde{\varphi}_0$ a lift of φ_0. Then $\exists! \tilde{H}$ s.t.

$$\begin{tikzcd} B \times 0 \arrow[r, "\tilde{\varphi}_0"] \arrow[d] & \mathbb{R} \arrow[d, "\varepsilon"] \\ B \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & S^1 \end{tikzcd}$$

If H is stationary on some $A \subseteq B$, then so is $\tilde{H}$.

(Proofs deferred)

Cor 1 (Path lifting for S^1) If $f : I \to S^1$, $r_0 \in \varepsilon^{-1}\{f(0)\}$, then $\exists!$ lift $\tilde{f} : I \to \mathbb{R}$ of f s.t. $\tilde{f}(0) = r_0$; any other lift of f takes the form $\tilde{f} + n$ for some $n \in \mathbb{Z}$.

Pf Apply II to $B = *$, $H = f$, $\tilde{\varphi}_0 = r_0$ to produce $\tilde{f}$. If $\tilde{f}'$ is some

other lift, then $\varepsilon(\tilde f(s)) = \varepsilon(\tilde f'(s)) \Rightarrow \tilde f(s) - \tilde f'(s) \in \mathbb{Z} \ \forall s$.
Since I is connected, $\tilde f - \tilde f'$ cts, know $\tilde f - \tilde f'$ is constant. $\square$

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Cor 2 (Path lifting criterion for S^1) Suppose $f_0, f_1 : I \to S^1$ with same endpoints, and $\tilde f_0, \tilde f_1$ are lifts w/ same initial point. Then $f_0 \sim f_1$ iff $\tilde f_0(1) = \tilde f_1(1)$.

I.e. $f_0 \sim f_1$ iff they have the same net angular change!

Pf If $\tilde f_0, \tilde f_1$ have the same terminal point then they are path htpic since $\mathbb{R}$ is simply conn'd. Thus $f_0 = \varepsilon \tilde f_0$, $f_1 = \varepsilon \tilde f_1$ are path htpic.

Now suppose $H : f_0 \sim f_1$. By htpy lifting,

$$\begin{tikzcd} I \times 0 \arrow[r, "\tilde f_0"] \arrow[d] & \mathbb{R} \arrow[d, "\varepsilon"] \\ I \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde H"] & S^1 \end{tikzcd}$$

and $\tilde H : \tilde f_0 \sim \underbrace{\tilde H(-,1)}_{\text{lift of } f_1 \text{ starting at } \tilde f_0(1)}$.