Prop If $f: X \simeq Y$, then $\tilde{X}$ and $\tilde{Y}$ are deformation retracts of $Cyl(f)$.

Thus two spaces are htpy equiv iff they are deformation retracts of a common space.

Pf Read 7.46. $\square$

8. XI. 22

Higher homotopy groups

π_0 — path components

π_1 — fundamental group

$\pi_n, n>1$ — higher htpy groups.

π_1 detects holes with loops $[S^1 \to X]$

π_n detects higher dimensional holes with $[S^n \to X]$

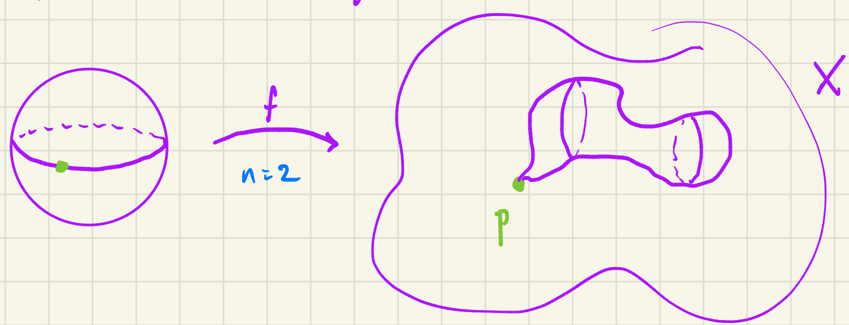
Given $p \in X$, $q \in Y$ write $\underbrace{f: \underbrace{(X,p)}_{\text{based space}} \longrightarrow (Y,q)}_{\text{based map}}$ for a cts map f s.t. $f(p) = q$

A based homotopy of based maps $f, g : (X, p) \to (Y, q)$ is a htpy $H : f \simeq g$ s.t. $H(p,t) = q \ \forall t \in [0,1]$.

Note that $\pi_1(X,p) = \{$path htpy classes of loops based at $p\}$
$= \{$based htpy classes of based maps $(S^1, (1,0)) \to (X,p)\}$.

For $n \geq 0$, set

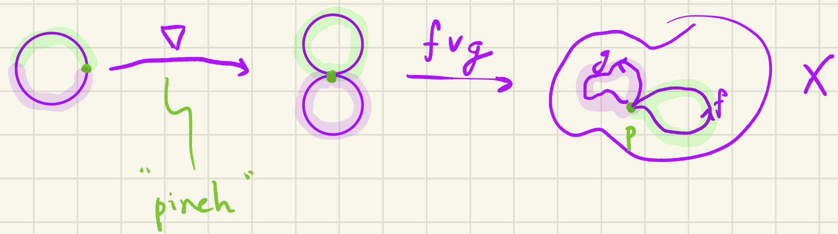
$$\pi_n(X,p) := \{\text{based htpy classes of based maps } (S^n, e_1) \to (X,p)\}.$$



Note $n=1$: retrieve old $\pi_1(X,p)$
$n=0$: $S^0 = \{\pm 1\}$, $f(1) = p$, $f(-1)$ arbitrary in X, $f \sim g$ iff $\exists$ path $f(-1) \rightsquigarrow g(-1)$.

We gave $\pi_1(X,p)$ a group structure via concatenation of loops.

On circle representatives, this is equivalent to

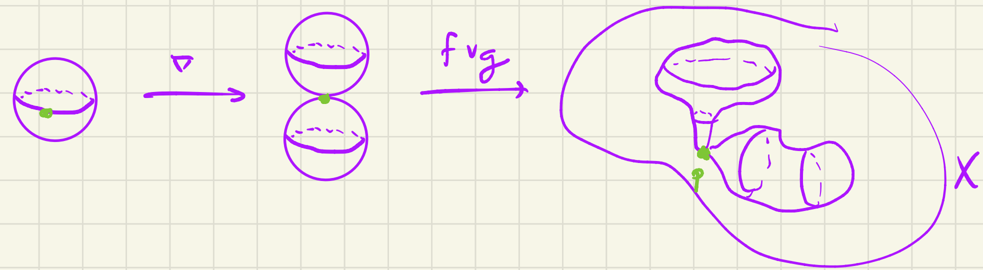


We can do the same thing in higher dimensions using the pinch map

$$\nabla : S^n \longrightarrow S^n / \underbrace{\{x \in S^n \mid x_{n+1} = 0\}}_{\text{"equator"}} \cong S^n \vee S^n .$$

Given $f, g : (S^n, e_1) \longrightarrow (X, p)$, define $[f] + [g] := [(f \vee g) \circ \nabla]$

(and check it's well-defined).



Prop For $n \geq 1$, $\pi_n(X,p)$ is a group under $+$ with identity $[\mathrm{const}_p]$.

Pf

$$\boxed{f \mid c_p} \sim \boxed{f} \sim \boxed{c_p \mid f}$$

$$\bar{f} = (\text{reflect } f): x \longmapsto f(e, -x)$$

$$\boxed{f \mid \bar{f}} \sim \boxed{c_p} \sim \boxed{\bar{f} \mid f}$$

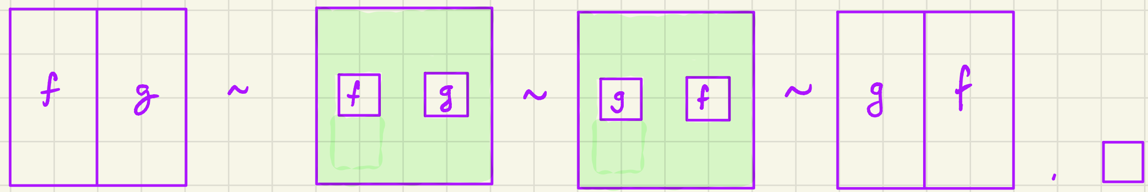
$$\boxed{f \mid g \mid h} \sim \boxed{f \mid g \mid h}$$

Q What about $n=0$?

$$S^0 \xrightarrow{?} S^0 \vee S^0$$

Thm For $n \geq 2$, $\pi_n(X, p)$ is an Abelian group.

Pf



Note $\varphi : (X, p) \longrightarrow (Y, q)$ induces $\varphi_* : \pi_n(X, p) \longrightarrow \pi_n(Y, q)$

$$[f] \longmapsto [\varphi f]$$

- φ_* is a group hom
- $(\psi\varphi)_* = \psi_* \varphi_*$
- $id_* = id$

$\Bigg\}$ π_n is a functor $Top_* \longrightarrow Gp$

AbGp for $n \geq 2$

Fact π_n factors through Hot_*

(category of based spaces + based htpy classes of based maps)

Categories + Functors

A category C consists of

- $Ob(C)$ — class of objects
- For $x, y \in Ob(C)$, class of morphisms $C(\underset{\text{source}}{x}, \underset{\text{target}}{y})$
- For $x, y, z \in Ob(C)$, composition
$$C(x,y) \times C(y,z) \longrightarrow C(x,z)$$
$$(f,g) \longmapsto gf$$

satisfying

- associativity of composition $(fg)h = f(gh)$
- identity: for $x \in Ob(C)$ $\exists\, id_x \in C(x,x)$ s.t. $id_y f = f = f\, id_x$ for all $f : x \to y$ (i.e. $f \in C(x,y)$).

E.g. **Set** : sets + functions

Gp : groups + homomorphisms

Ab : Abelian groups + homomorphisms

Top : spaces + cts functions

Hot : spaces + htpy classes of functions

also $\mathbf{Top}_*$, $\mathbf{Hot}_*$ — based versions

$\mathbf{Mat}_k$: $\mathbb{N}$ + $\mathrm{Mat}_k(n,m) = \{ m\times n \text{ matrices w/ entries in } k \}$

composition = matrix mult'n! — related to

$\mathbf{FinVec}_k$: fin dim'l k-vector spaces + linear transformations

Defn A (covariant) functor $F: \underbrace{C \to D}_{\text{categories}}$ is assignments

$F: \mathrm{Ob}(C) \to \mathrm{Ob}(D)$ + $F: C(x,y) \to D(Fx, Fy)$ s.t.

- $F(gh) = (Fg)(Fh)$ and
- $F(\mathrm{id}_x) = \mathrm{id}_{Fx}$.

A contravariant functor is the same except $F : \mathcal{C}(x, y) \to \mathcal{D}(Fy, Fx)$ order swapped! and $F(gh) = (Fh)(Fg)$. Write $F : \mathcal{C}^{op} \to \mathcal{D}$.
opposite category

E.g.
- $\pi_1 : \mathbf{Top}_* \to \mathbf{Gp}$ or $\pi_1 : \mathbf{Hot}_* \to \mathbf{Gp}$
- $\pi_n : \mathbf{Top}_*$ or $\mathbf{Hot}_* \to \mathbf{Ab}$ for $n \geq 2$
- $\pi_0 : \mathbf{Top} \to \mathbf{Set}$ or $\mathbf{Hot} \to \mathbf{Set}$
- Forgetful functors $U : \mathbf{Top} \to \mathbf{Set}$, $U : \mathbf{Gp} \to \mathbf{Set}$, $U : \mathbf{CRing} \to \mathbf{Ring}$, ...
- $2^{(\,)} : \mathbf{Set}^{op} \to \mathbf{Set}$

$$\begin{tikzcd} X \arrow[d, "f"'] & & 2^X \\ Y \arrow[r, phantom, "\longmapsto"] & & 2^Y \arrow[u, "f^{-1}"'] \end{tikzcd}$$

- $C : \mathbf{Top}^{op} \to \mathbf{CRing}$ cts fns $X \to \mathbb{R}$

$$\begin{tikzcd} X \arrow[d, "f"'] & & C(X) & \varphi f \\ Y \arrow[r, phantom, "\longmapsto"] & & C(Y) \arrow[u] & \varphi \arrow[u, mapsto] \end{tikzcd}$$

An **isomorphism** in a cat C is $f: x \to y \in C(x,y)$ for which $\exists\, g: y \to x$ s.t. $gf = id_x$, $fg = id_y$.

Thm If $F: C \to D$ or $F: C^{op} \to D$ is a functor, and φ is an isomorphism, then $F\varphi$ is an isomorphism.

Pf

$$F\left(\begin{tikzcd} & y \arrow[rr, "id_y"] \arrow[dr, "\psi"] & & y \\ x \arrow[ur, "\varphi"] \arrow[rr, "id_x"'] & & x \arrow[ur, "\varphi"'] & \end{tikzcd}\right) = \begin{tikzcd} & Fy \arrow[rr, "id_{Fy}"] \arrow[dr, "F\psi"] & & Fy \\ Fx \arrow[ur, "F\varphi"] \arrow[rr, "id_{Fx}"'] & & Fx \arrow[ur, "F\varphi"'] & \end{tikzcd}$$

$\square$

More examples

- $\mathbb{N}$, unique $m \to n$ iff $m \mid n$; $Fib: \mathbb{N} \to \mathbb{N}$
- Fundamental groupoid
- Cobordism / tangle cats

$\mathrm{ob} : \mathbb{N}$

$$\mathbb{N}(m,n) = \begin{cases} * & m \mid n \\ \varnothing & m \nmid n \end{cases}$$

$\mathrm{Fib} : \mathbb{N} \longrightarrow \mathbb{N}$ $\qquad$ $\mathrm{Fib}_0 = 0,\ \mathrm{Fib}_1 = 1,\ \mathrm{Fib}_{n+1} = \mathrm{Fib}_n + \mathrm{Fib}_{n-1}$

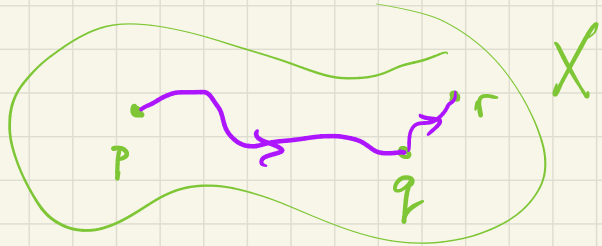
$$\begin{tikzcd} n \arrow[r, mapsto] \arrow[d] & \mathrm{Fib}_n \arrow[d] \\ m \arrow[r, mapsto] & \mathrm{Fib}_m \end{tikzcd}$$

indeed, $n \mid m \Rightarrow \mathrm{Fib}_n \mid \mathrm{Fib}_m$

$\left(\mathrm{Fib}_{\gcd(m,n)} = \gcd(\mathrm{Fib}_m, \mathrm{Fib}_n) \right)$

space X, $\Pi_1 X$: pts of X + $\Pi_1 X(p,q)$

= path htpy classes of paths in X from p to q



<u>Note</u> Every morphism in $\Pi_1 X$ is an isomorphism
inverse of $[f]$ is $[\bar{f}]$

$$\Pi_1 X(p,p) = \pi_1(X,p)$$