Prop If $r : X \to A$ is a retraction, then $\forall p \in A$, $(\iota_A)_* : \pi_1(A,p) \to \pi_1(X,p)$ is injective and $r_* : \pi_1(X,p) \to \pi_1(A,p)$ is surjective.

Pf
$$\begin{tikzcd} & X \arrow[dr, "r"] & \\ A \arrow[ur, "\iota_A"] \arrow[rr, "\mathrm{id}_A"'] & & A \end{tikzcd} \quad \text{induces} \quad \begin{tikzcd} & \pi_1(X,p) \arrow[dr, "r_*"] & \\ \pi_1(A,p) \arrow[ur, "(\iota_A)_*"] \arrow[rr, "\mathrm{id}"'] & & \pi_1(A,p) \end{tikzcd} . \qquad \square$$

Cor A retract of a simply conn'd space is simply conn'd. $\square$
(A a retract of X, $\pi_1(X,p) = e \Rightarrow \pi_1(A,p) = e$.)

E.g. $\pi_1(S^1, 1) \cong \mathbb{Z} \Rightarrow \mathbb{R}^2 \setminus \{0\}$ is not simply conn'd $\Rightarrow \mathbb{R}^2 \setminus 0 \not\cong \mathbb{R}^2$.

E.g. $S^1 \times \{1\}$ is a retract of $T^2 = S^1 \times S^1$ via $(z,w) \mapsto (z,1)$.
Thus T^2 is not simply conn'd and not $\cong S^2$.

7.XI.22

π_1(products) Write $p_i : X_1 \times \cdots \times X_n \to X_i$ for i-th projection.
Given basepoints $x_i \in X_i$, get $(p_i)_* : \pi_1(X_1 \times \cdots \times X_n, (x_1, \ldots, x_n)) \to \pi_1(X_i, x_i)$.

By the universal property of products, we get

$$\begin{aligned} P : \pi_1(X_1 \times \cdots \times X_n, (x_1, \ldots, x_n)) &\longrightarrow \prod_{i=1}^{n} \pi_1(X_i, x_i) \\ [f] &\longmapsto ((p_1)_*[f], \ldots, (p_n)_*[f]). \end{aligned}$$

<u>Prop</u> P is an iso of groups.

<u>Pf</u> Since each $(p_i)_*$ is a homomorphism, P is a homomorphism.

- For surjectivity, choose $[f_i] \in \pi_1(X_i, x_i)$ for $1 \le i \le n$. Define
$$\begin{aligned} f : I &\longrightarrow X_1 \times \cdots \times X_n \\ s &\longmapsto (f_1(s), \ldots, f_n(s)). \end{aligned}$$
Since $f_i = p_i f$, we get
$$P[f] = ([f_1], \ldots, [f_n]).$$

- For injectivity, suppose $P[f] = ([c_{x_1}], \ldots, [c_{x_n}])$. Choose htpies $H_i : f_i \sim c_{x_i}$. Then $H = (H_1, \ldots, H_n) : f \sim [c_{(x_1, \ldots, x_n)}]$. $\square$

Homotopy equivalence (All maps cts unless explicitly not)

Homeomorphism:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "\mathrm{id}_X"'] & Y \arrow[d, "g"] \arrow[dr, "\mathrm{id}_Y"] & \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Homotopy equivalence:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "\mathrm{id}_X"'] & Y \arrow[d, "g"] \arrow[dr, "\mathrm{id}_Y"] \arrow[dl, phantom, "\Swarrow"] \arrow[dr, phantom, "\Nearrow"'] & \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

i.e. $\exists g : Y \longrightarrow X$ s.t. $gf \simeq \mathrm{id}_X$, $fg \simeq \mathrm{id}_Y$.

When a htpy equivalence $f : X \to Y$ exists, write $f : X \simeq Y$ and call X, Y **homotopy equivalent**.

Prop $\simeq$ is an equivalence relation on the class of topological spaces.

Defn A *deformation retraction* is a retraction $r : X \to A$ (so $r\iota_A = \mathrm{id}_A$) such that $\iota_A r \simeq \mathrm{id}_X$. Call A a *deformation retract* of X.

In this case, $r : X \simeq A : \iota_A$.

Unpacking: $\exists$ htpy $H : X \times I \to X$ s.t.

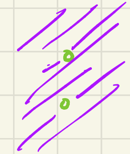
$$H(x,0) = x \quad \forall x \in X$$

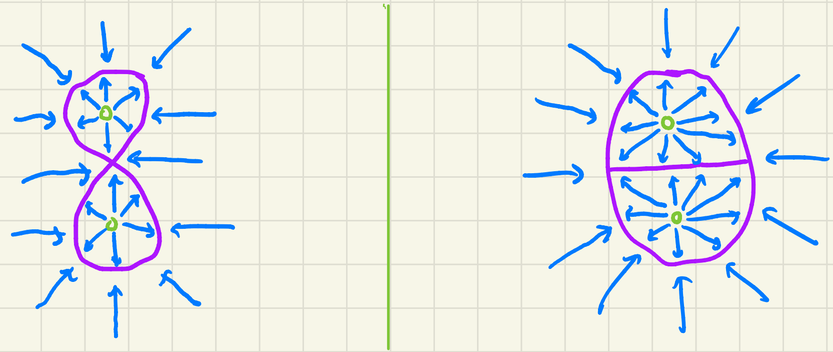
$$H(x,1) \in A \quad \forall x \in X$$

$$H(a,1) = a \quad \forall a \in A$$

(if $H(a,t) = a$ $\forall a \in A, t \in I$, have a *strong deformation retract*)

E.g. For $n \geq 1$, S^{n-1} is a strong deformation retract of $\mathbb{R}^n \setminus 0$ and of $\bar{\mathbb{B}}^n \setminus 0$. Indeed, take $H(x,t) = (1-t)x + t\frac{x}{|x|}$.

E.g. 8 and Θ are both strong deformation retracts of  :



Thus $8 \simeq \Theta$.

Thm If $\varphi : X \simeq Y$, then $\varphi_* : \pi_1(X,p) \cong \pi_1(Y, \varphi(p))$

(!) $\psi\varphi \simeq id_X$ is not enough to know $\psi\varphi(p) = id_X(p) = p$.

Lemma Suppose $\varphi, \psi : X \longrightarrow Y$, $H : \varphi \simeq \psi$. Fix $p \in X$ and let $h : I \longrightarrow Y$, $t \longmapsto H(p,t)$

(a path from $\varphi(p)$ to $\psi(p)$), and let $\Phi_h : \pi_1(Y, \varphi(p)) \xrightarrow{\cong} \pi_1(Y, \psi(p))$

$[f] \longmapsto [\bar{h}][f][h]$.

Then the following diagram commutes:

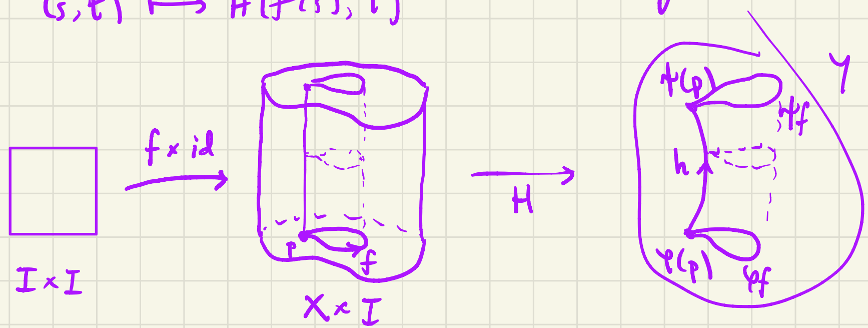
$$\begin{tikzcd} & \pi_1(Y, \varphi(p)) \arrow[dd, "\Phi_h"] \\ \pi_1(X,p) \arrow[ur, "\varphi_*"] \arrow[dr, "\psi_*"'] & \\ & \pi_1(Y, \psi(p)) \end{tikzcd}$$

.

Pf Take $[f] \in \pi_1(X,p)$. WTS $\psi_*[f] = \Phi_h(\varphi_*[f])$, i.e. $\psi f \sim \bar{h}\cdot(\varphi f)\cdot h$ which is the case iff $h \cdot \psi f \sim \varphi f \cdot h$. Consider

$$F : I \times I \longrightarrow Y$$
$$(s,t) \longmapsto H(f(s), t)$$

and apply the square lemma:



□

Pf of Thm Suppose $\varphi : X \simeq Y : \psi$. Consider

$$\begin{tikzcd}
\pi_1(X,p) \arrow[r, "\varphi_*"] \arrow[rr, bend right, "\cong"', "\text{Lemma}", "\Phi_h"'] & \pi_1(Y,\varphi(p)) \arrow[r, "\psi_*"] \arrow[rr, bend left, "\cong", "\text{Lemma}"', "\Phi_k"] & \pi_1(Y,\psi(\varphi(p))) \arrow[r, "\varphi_*"] & \pi_1(Y,\varphi(\psi(\varphi(p))))
\end{tikzcd}$$

Since $\psi_*\varphi_*$ is an iso, φ_* is injective and ψ_* is surjective.

Similarly, $\varphi_*\psi_*$ is an iso, so ψ_* is injective. Thus ψ_* is an iso, whence

$\varphi_* = (\psi_*)^{-1} \circ \Phi_h$ is a composite of isos, hence iso. $\square$

Mapping Cylinders

Given $f : X \to Y$, define $Cyl(f) := Y \cup_\varphi (X \times I)$ for $\varphi : X \times 0 \to Y$, $(x,0) \mapsto f(x)$:

Note $\tilde{X} \cong X$, $\tilde{Y} \cong Y$.

Prop If $f: X \simeq Y$, then $\tilde{X}$ and $\tilde{Y}$ are deformation retracts of $\mathrm{Cyl}(f)$.

Thus two spaces are htpy equiv iff they are deformation retracts of a common space.

Pf Read 7.46. $\square$