$$= [\bar{g}][f_1][f_2][g]$$

$$= \Phi_g([f_1][f_2]).$$

Since $\Phi_{\bar{g}}$ is inverse to Φ_g, it's an isomorphism. $\square$

4. XI. 22

<u>Circle Representatives</u>

A loop $f: I \to X$ is the same thing as $\hat{f}: S^1 \to X$.
$0,1 \mapsto p$ $\qquad$ $\exp(2\pi i t) \mapsto f(t)$
$1 \mapsto p$

<u>Prop</u> Suppose f is a loop based at p with circle rep $\tilde{f}$. Then TFAE

(a) $f \sim c_p$

(b) $\tilde{f} \simeq \text{const}$

(c)
$$\begin{tikzcd} S^1 \arrow[r, "\tilde{f}"] \arrow[d, hook] & X \\ D^2 \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Pf (a) $\Rightarrow$ (b): Take $H: f \simeq c_p$. Then

$$\begin{tikzcd} I \times I \arrow[r, "H"] \arrow[d] & X \\ S^1 \times I \arrow[ur, dashed, "\tilde{H}"'] & \end{tikzcd}$$

respects vertical map's identifications

quotient map by closed map lemma

Then $\tilde{H}: \tilde{f} \simeq c_p$.

(b) $\Rightarrow$ (c): Suppose $H: S^1 \times I \to X$ is a htpy b/w $\tilde{f}$ and a constant map $k: S^1 \to X$. Then

$$\begin{tikzcd} S^1 \times I \arrow[r, "H"] \arrow[d] & X \\ D^2 \cong CS^1 \arrow[ur, dashed, "\tilde{H}"'] & \end{tikzcd}$$

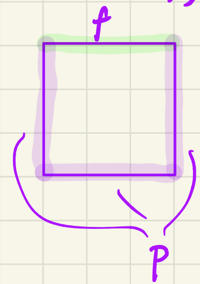
respects vertical quotient map's identifications.

and $\tilde{H}|_{S^1} = \tilde{f}$.

CS^1

(c) $\Rightarrow$ (a): Assume $\tilde{f}$ extends to $F: D^2 \to X$. Since D^2 is convex, have the straight line htpy $H: c_1 \simeq \omega$ for $\omega: I \to S^1 \subseteq D^2$, $t \mapsto \exp(2\pi i t)$.

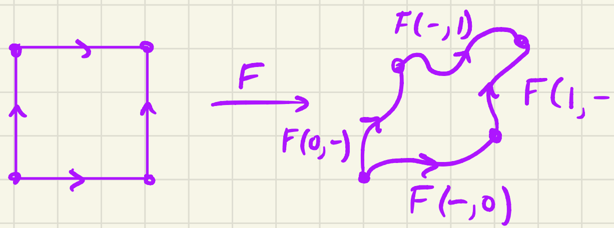
Now $H(\partial(I\times I)) \subseteq S^1$, so $FH: I\times I \longrightarrow X$ satisfies:



thus $FH: c_p \sim f$. $\square$

Cor (Square lemma) If $F: I\times I \longrightarrow X$ is a cts map, then

$$F(-,0)\cdot F(1,-) \sim F(0,-)\cdot F(-,1).$$



Pf Previous prop + $f\sim g \iff f\bar{g} \sim \text{const}$. $\square$

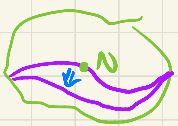
π_1 (spheres)

Defn Call a space X simply connected when it is path connected and $\pi_1(X, p)$ is trivial for some (and hence all) $p \in X$.

Thm For $n \geq 2$, S^n is simply connected.

Pf Sketch Choose a base point $p \neq N = (0, \ldots, 0, 1) \in S^n$. If $f : I \to S^n$ is a ~~path~~ loop based at p not passing through N, then f is a loop in $S^n \setminus \{N\} \cong \mathbb{R}^n$ and thus is null-htpic (via, e.g., straight line htpy). If f does pass through N, "nudge" it so it (a) is $\simeq f$ and (b) doesn't pass through N. Proceed as before. $\square$

⚠ The "nudging" is delicate! Use Lebesgue number lemma to guarantee you can do it. (pp. 194-195)



π_1(manifolds)

Thm The fundamental group of a manifold is countable.

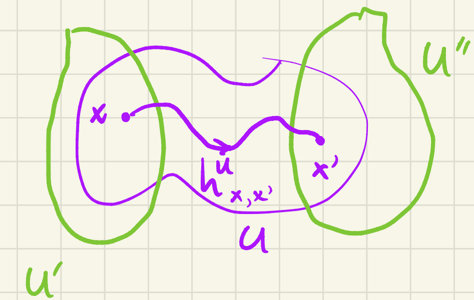
Pf (sketch) M a mfld, $\mathcal{U}$ a countable cover of M by coordinate balls.

For each $U, U' \in \mathcal{U}$, $U \cap U'$ has countably many components.

Choose a point in each such component (as $U, U' \in \mathcal{U}$ vary) and let $\mathcal{X}$ denote the countable set of such points.

For $U \in \mathcal{U}$ and $x, x' \in \mathcal{X}$ s.t. $x, x' \in U$, choose a path $h^U_{x,x'} : x \rightsquigarrow x'$ in U.

Choose some $p \in \mathcal{X}$ as base point. Call a loop based at p special when it is a finite product of paths of the form $h^U_{x,x'}$.

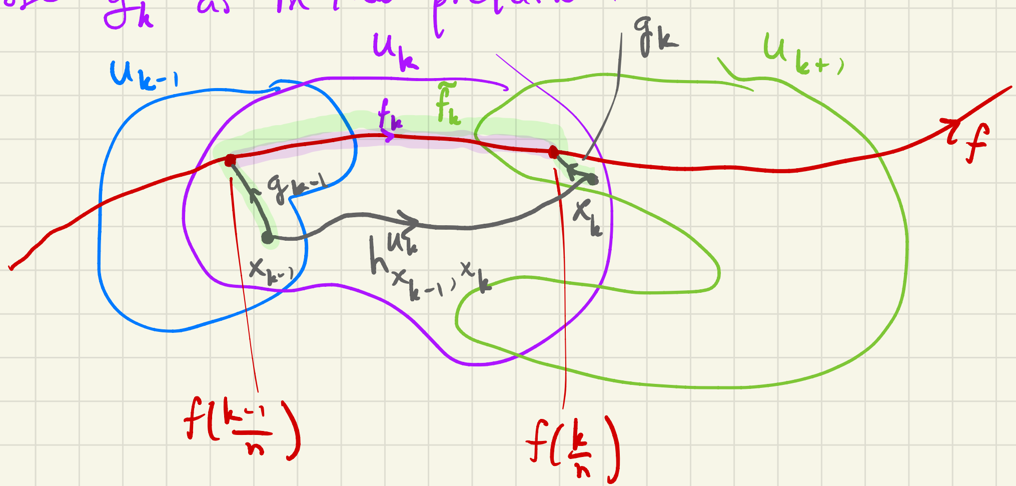


Since $\mathcal{U} \times \mathcal{X}^2$ is countable, it suffices to show that every loop based at p

is path htpic to a special path.

By the Lebesgue number lemma (open covers of compact metric spaces have a Lebesgue number: $\delta > 0$ s.t. every set w/ diameter $< \delta$ is in some $U \in \mathcal{U}$), can produce $n \in \mathbb{Z}$ s.t. $f\left[\frac{k-1}{n}, \frac{k}{n}\right]$ is a subset of some $U_k \in \mathcal{U}$ for each $1 \leq k \leq n$. Let $f_k := f|_{\left[\frac{k-1}{n}, \frac{k}{n}\right]}$ and note $f \sim f_1 \cdots f_n$.

Now choose g_k as in the picture:



Set $\tilde{f}_k := g_{k-1} \cdot f_k \cdot \bar{g}_k$. Then $f \sim \tilde{f}_1 \cdots \tilde{f}_n$ b/c $\bar{g}_k g_k$ cancel.

Furthermore, $\tilde{f}_k \sim h^{u_k}_{x_{k-1}, x_k} \implies f \sim$ special path. $\square$

π_1 is a functor

Prop If $f_0 \sim f_1 : I \to X$ and $\varphi : X \to Y$ is cts, then $\varphi f_0 \sim \varphi f_1$. $\square$

As such, φ induces a well-defined map
$$\begin{aligned} \varphi_* : \pi_1(X, p) &\longrightarrow \pi_1(Y, \varphi(p)) \\ [f] &\longmapsto [\varphi f]. \end{aligned}$$

Prop For any cts φ, φ_* is a group homomorphism.

Pf We have $\varphi_*([f][g]) = \varphi_*[f \cdot g] = [\varphi \circ (f \cdot g)]$. But

$\varphi \circ (f \cdot g) = (\varphi \circ f) \cdot (\varphi \circ g)$ on the nose! $\square$

Prop (a) If $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ are cts, then $(\psi\varphi)_* = \psi_* \varphi_*$.

(b) $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,p)}$.

Pf (a) $(\psi\varphi)_*[f] = [(\psi\varphi)f] = [\psi(\varphi f)] = \psi_*[\varphi f] = \psi_*(\varphi_*[f])$

(b) $(\mathrm{id}_X)_*[f] = [\mathrm{id}_X f] = [f]$. $\square$

Cor If $\varphi : X \xrightarrow{\cong} Y$, then $\varphi_* : \pi_1(X,p) \xrightarrow{\cong} \pi_1(Y, \varphi(p))$.

Pf Check that φ_* and $(\varphi^{-1})_*$ are inverses. $\square$

$S^1 \hookrightarrow \mathbb{R}^2$ induces $\mathbb{Z} \to e$ on π_1.

For $A \subseteq X$,

Defn a map $r : X \to A$ is a *retraction* if $r|_A = \mathrm{id}_A$ (or equivalently, $r\iota_A = \mathrm{id}_A$). If $\exists$ retraction $X \to A$, call A a *retract* of X.

E.g. $\mathbb{R}^n \setminus 0 \to S^{n-1}$, $x \longmapsto \frac{x}{\|x\|}$ is a retraction.

__Prop__ If $r: X \to A$ is a retraction, then $\forall p \in A$, $(\iota_A)_* : \pi_1(A,p) \to \pi_1(X,p)$ is injective and $r_* : \pi_1(X,p) \to \pi_1(A,p)$ is surjective.

__Pf__

$$\begin{tikzcd} & X \arrow[dr, "r"] & \\ A \arrow[ur, "\iota_A"] \arrow[rr, "\mathrm{id}_A"'] & & A \end{tikzcd} \quad \text{induces} \quad \begin{tikzcd} & \pi_1(X,p) \arrow[dr, "r_*"] & \\ \pi_1(A,p) \arrow[ur, "(\iota_A)_*"] \arrow[rr, "\mathrm{id}"'] & & \pi_1(A,p) \end{tikzcd}.$$ $\square$

__Cor__ A retract of a simply conn'd space is simply conn'd. $\square$
(A a retract of X, $\pi_1(X,p)=e \Rightarrow \pi_1(A,p)=e$.)

__E.g.__ $\pi_1(S^1,1) \cong \mathbb{Z} \Rightarrow \mathbb{R}^2 \setminus \{0\}$ is not simply conn'd $\Rightarrow \mathbb{R}^2 \setminus 0 \not\cong \mathbb{R}^2$.

__E.g.__ $S^1 \times \{1\}$ is a retract of $T^2 = S^1 \times S^1$ via $(z,w) \mapsto (z,1)$.
Thus T^2 is not simply conn'd and not $\cong S^2$.

7.XI.22

__π_1(products)__ Write $p_i : X_1 \times \cdots \times X_n \to X_i$ for i-th projection.
Given basepoints $x_i \in X_i$, get $(p_i)_* : \pi_1(X_1 \times \cdots \times X_n, (x_1, \dots, x_n)) \to \pi_1(X_i, x_i)$.