1.XI.22

Homotopy

Let $I = [0,1]$. Given cts maps $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, a **homotopy**

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow H"] & Y \end{tikzcd}$$

from f to g (also written $H : f \Rightarrow g$ or $H : f \simeq g$)

is a cts function $H : X \times I \longrightarrow Y$ s.t.

$$\begin{tikzcd} X \times \{0\} \arrow[d] \arrow[dr, "f"] & \\ X \times I \arrow[r, "H"'] & Y \\ X \times \{1\} \arrow[u] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes

i.e. $H(x,0) = f(x)$, $H(x,1) = g(x)$.

$\therefore$ $\{$ H is a "movie" starting at $t=0$ with f, ending at $t=1$ with g.

Prop $\simeq$ is an equivalence relation on $\mathrm{Top}(X,Y) := \{ f \in Y^X \mid f \text{ cts} \}$.

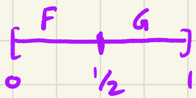
Pf Reflexive: $H(x,t) = f(x)$ $\forall t$

Symmetric: Given $H: f \Rightarrow g$, define $\bar{H}(x,t) = H(x, 1-t)$.

Then $\bar{H}: g \Rightarrow f$.

Transitive: Given
$$\begin{tikzcd} X \arrow[r, bend left=50, "f"{name=A}] \arrow[r, "g"{name=B}] \arrow[r, bend right=70, "h"'{name=C}] & Y \arrow[from=A, to=B, Rightarrow, shorten=2pt, "F"'] \arrow[from=B, to=C, Rightarrow, shorten=2pt, "G"'] \end{tikzcd}$$
define
$$H(x,t) = \begin{cases} F(x, 2t) & 0 \le t \le \frac{1}{2} \\ G(x, 2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

i.e.



. Then $H: f \Rightarrow h$. $\square$

Prop If
$$\begin{tikzcd} X \arrow[r, bend left=50, "f_0"{name=A}] \arrow[r, bend right=50, "f_1"'{name=B}] & Y \arrow[r, bend left=50, "g_0"{name=C}] \arrow[r, bend right=50, "g_1"'{name=D}] & Z \arrow[from=A, to=B, Rightarrow, shorten=2pt, "F"'] \arrow[from=C, to=D, Rightarrow, shorten=2pt, "G"'] \end{tikzcd}$$
then $g_0 f_0 \simeq g_1 f_1$.

Pf Define $H(x,t) = G(F(x,t), t)$. Then
$$\begin{aligned} H(x,0) &= G(F(x,0), 0) \\ &= G(f_0(x), 0) \\ &= g_0(f_0(x)) \end{aligned}$$
and similarly $H(x,1) = g_1(f_1(x))$ so $H: g_0 f_0 \Rightarrow g_1 f_1$. $\square$

E.g. For $f, g : \mathbb{R} \to \mathbb{R}^2$ given by $f(x) = (x, x^2)$, $g(x) = (x, x)$, we have $H(x,t) = (x, x^2 - tx^2 + tx)$ a htpy from f to g.

e.g.: $f(t) = (\cos(2\pi t), \sin(2\pi t))$
$g(t) = f(2t)$
$0 \le t \le 1$

E.g. For $f, g : X \to B \subseteq \mathbb{R}^n$ convex, we have the straight line htpy $H(x,t) = (1-t) f(x) + t\, g(x)$ from f to g.

For $A \subseteq X \underset{g}{\overset{f}{\rightrightarrows}} Y$ call $H : f \Rightarrow g$ stationary on A when $H(x,t) = f(x)$ $\forall x \in A, t \in I$. Then call f, g homotopic relative to A.

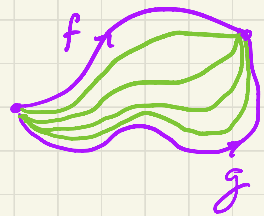
Note $f|_A = H(-,1)|_A = g|_A$ so f, g agree on A in this case.

The Fundamental Group

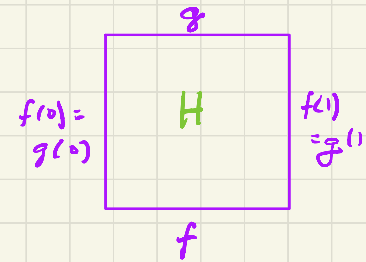
Idea: Use loops to detect "holes".

Recall that a path in X is a cts function $f : I \to X$.

Given paths f, g in X, a **path homotopy** from f to g is $H : f \Rightarrow g$ stationary on $\{0,1\}$:



or

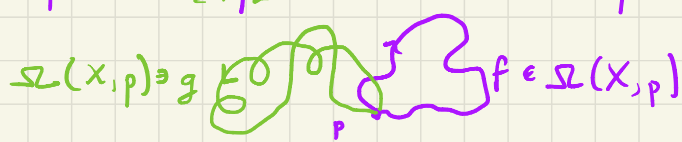


Call f **path homotopic** to g and write $f \sim g$.

Prop Path homotopy is an equivalence rel'n on paths from p to q in X.

Pf Check that previous constructions respect endpoints. $\square$

- Write $[f]$ for the **path class** of $f : I \to X$, i.e. the equiv class of f up to path htpy equiv.
- Write $\Omega(X,p)$ for **loops** in X based at p i.e. paths from p to p.



- $c_p : I \longrightarrow X$ is the constant loop at p
  $t \longmapsto p$

- A null-homotopic loop if $f \sim c_p$.

Lemma Any $\underbrace{\text{reparametrization}}$ of a path f is path-htpic to f.

$\varphi : I \to I$, $\varphi(0)=0$, $\varphi(1)=1$
$f\varphi : I \longrightarrow X$ is a reparametrization.

(e.g. $\varphi(s)=s^2$,
$f\varphi : t \longmapsto f(t^2)$)

Pf Take $H : I \times I \longrightarrow I$ the straight line htpy from id_I to φ.

$$(s,t) \longmapsto (1-t)s + t\varphi(s)$$

Then $fH : f \sim f\varphi$. $\square$

i.e. $fH(s,0) = f(s)$, $fH(s,1) = f\varphi(s)$
$fH(0,t) = f(t\varphi(0)) = f(0)$, $fH(1,t) = f(1)$.

Define $\pi_1(X,p) := \Omega(X,p)/\sim$ and endow it with the following binary operation:

Given $\underbrace{f : p \rightsquigarrow q}_{\text{path in } X \text{ from } p \text{ to } q}$, $g : q \rightsquigarrow r$, define $f \cdot g : I \longrightarrow X$

$$s \longmapsto \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

Since $f(1) = q = g(0)$, $f \cdot g : p \rightsquigarrow r$.

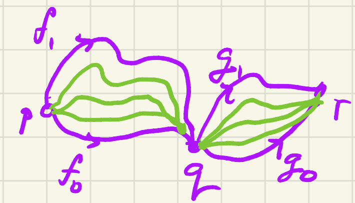
Prop Given $f_0 \sim f_1 : p \rightsquigarrow q$, $g_0 \sim g_1 : q \rightsquigarrow r$, we have

$$f_0 \cdot g_0 \sim f_1 \cdot g_1 : p \rightsquigarrow r.$$

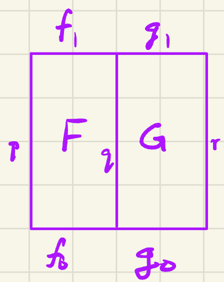
Pf If $F : f_0 \sim f_1$, $G : g_0 \sim g_1$, define

$$H : I \times I \longrightarrow X$$

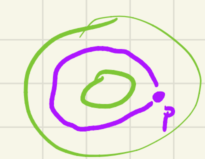
$$(s,t) \longmapsto \begin{cases} F(2s, t) & 0 \le s \le \frac{1}{2},\ 0 \le t \le 1 \\ G(2s-1, t) & \frac{1}{2} \le s \le 1,\ 0 \le t \le 1 \end{cases}$$



i.e. $H =$



Then $H : f_0 \cdot g_0 \sim f_1 \cdot g_1$. $\square$



Given f, g composable paths, define $[f] \cdot [g] := [f \cdot g]$.

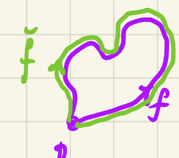
In particular, $\cdot$ is a binary operation on $\pi_1(X,p)$:

$$\cdot : \pi_1(X,p) \times \pi_1(X,p) \longrightarrow \pi_1(X,p)$$
$$([f],[g]) \longmapsto [f]\cdot[g] = [f\cdot g]$$

Claim This makes $\pi_1(X,p)$ a group!

Identity: $[c_p]$

Inverse: $[f]^{-1} = [\bar{f}]$ where $\bar{f}(t) = f(1-t)$



Thm For $f : p \rightsquigarrow q$, $g : q \rightsquigarrow r$, $h : r \rightsquigarrow s$ paths in X,

(a) $[c_p]\cdot[f] = [f] = [f]\cdot[c_q]$

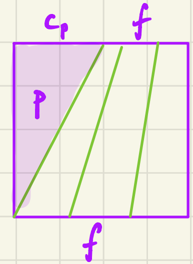
(b) $[f]\cdot[\bar{f}] = [c_p]$, $[\bar{f}][f] = [c_q]$

(c) $[f]\cdot([g][h]) = ([f]\cdot[g])\cdot[h]$.

Cor $(\pi_1(X,p), \cdot)$ is a group (and $\Pi_1 X := \mathrm{Top}(I,X)/\sim$ is a "groupoid").

Pf of Thm

(a)

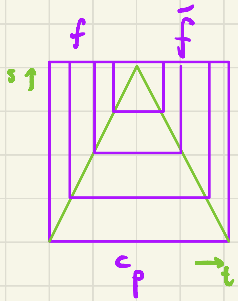


i.e. ~~reparametrization by~~ $\cancel{\varphi(t) = \begin{cases} p & 0 \le t \le \frac{1}{2} \\ 2t-1 & \frac{1}{2} \le t \le 1 \end{cases}}$

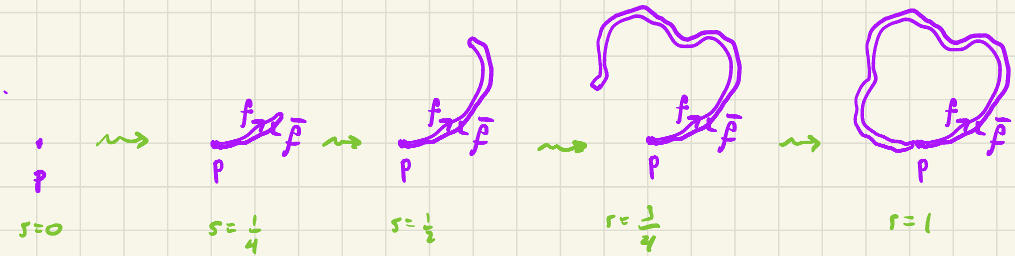
shows $f \sim c_p \cdot f$.

Similar for $f \sim f \cdot c_q$.

(b)



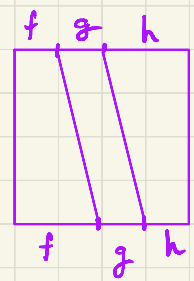
i.e.



so $f \cdot \bar{f} \sim c_p$. Swapping the roles of $f, \bar{f}$ and noting $\bar{\bar{f}} = f$, get $\bar{f} \cdot f \sim c_q$ as well.

(c)

$(f \cdot g) \cdot h$



$f \cdot (g \cdot h)$

$\square$

Note Choices in how to multiply loops lead to the theory of
$A_\infty = E_1$-spaces, operads, ...

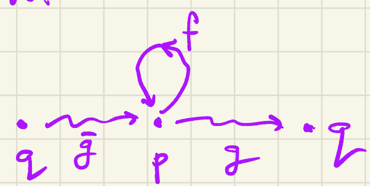
Thm Suppose X is path connected, $p, q \in X$, $g : p \rightsquigarrow q$. Then

$$\begin{aligned} \Phi_g : \pi_1(X,p) &\longrightarrow \pi_1(X,q) \\ [f] &\longmapsto [\bar{g}][f][g] \end{aligned}$$

is an isomorphism w/ inverse $\Phi_{\bar{g}}$.

Slogan Conjugate to change base points.

Pf First note Φ_g is well defined:



so $\Phi_g[f] \in \pi_1(X,q)$.

Next Φ_g is conjugation by $[g]$ hence a group homomorphism:

$$\Phi_g[f_1]\,\Phi_g[f_2] = [\bar{g}][f_1]\overbrace{[g][\bar{g}]}^{[c_p]}[f_2][g]$$

$$= [\bar{g}][f_1][f_2][\bar{g}]$$

$$= \Phi_g([f_1][f_2]).$$

Since $\Phi_{\bar{g}}$ is inverse to Φ_g, it's an isomorphism. $\square$