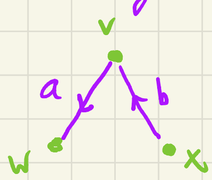
31. X. 22

Step 4 M admits a presentation in which all vxs are identified to a single point:

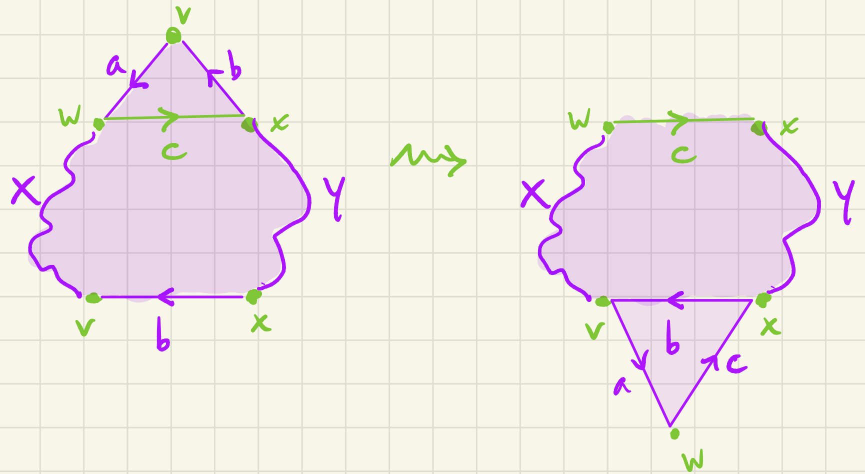
- Suppose there are more than one equivalence class of vxs after gluing. By induction, it suffices to eliminate an equivalence class (via elementary transformations) without changing the other properties of the presentation. By induction again, it suffices to eliminate at least one vx from a specified equiv class w/o changing other properties.

  Suppose v is a vx not equivalent to all other vxs. Then there is an edge a from v to a vx w not equivalent to v.



Have $b \neq a$ or a^{-1} by previous conditions.

Elsewhere have $v \xleftarrow{b} x$ or $v \xrightarrow{b} x$. Assume $v \xleftarrow{b} x$ (other case similar). Then the word in the presentation is of the form $baXb^{-1}Y$ after rotation:



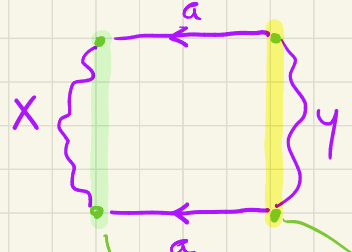
Now have fewer v vxs, more w vxs. Perform Step 2 to remove any new adjacent complementary pairs — doesn't increase # of v's, so we've succeeded.

Step 5 If the presentation has a complementary pair a, a^{-1}, then it has another complementary pair b, b^{-1} intertwined in its presentation: $a \cdots b \cdots a^{-1} \cdots b^{-1}$.

- If not, then the presentation is of the form $a\underbrace{X}a^{-1}Y$

only containing complementary pairs or adjacent twisted pairs.

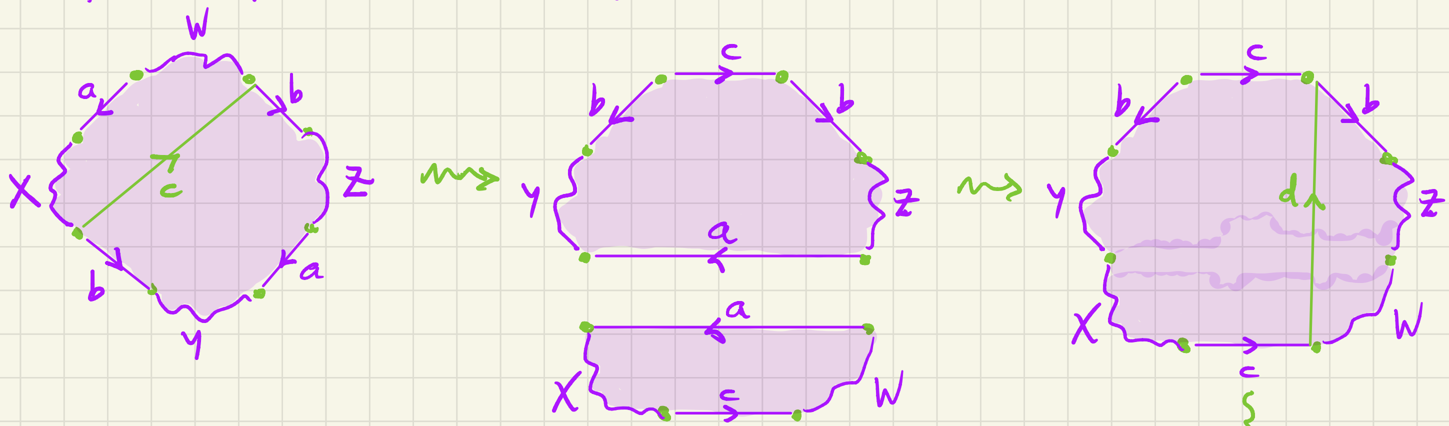
Thus edges of X are identified only w/ edges of X, and the same is true of Y:

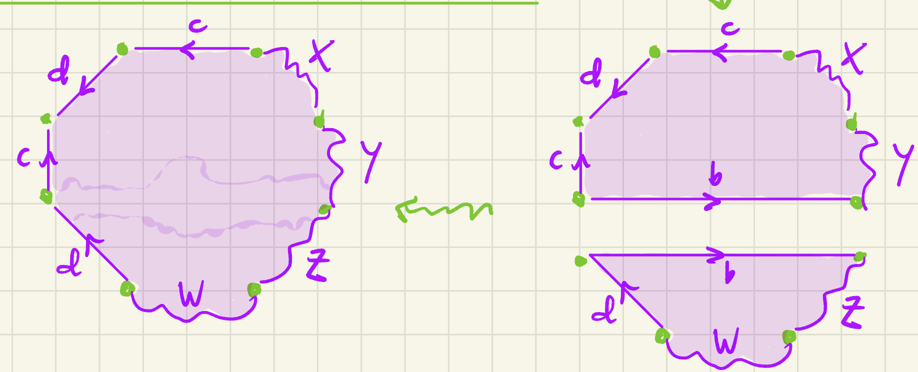


different vx equiv classes, ☹ ✓

Step 6 M admits a presentation in which all intertwined complementary pairs are adjacent: $aba^{-1}b^{-1}$.

- If the presentation is of the form $WaXbYa^{-1}Zb^{-1}$, then





This replaces $a \cdots b \cdots a^{-1} \cdots b^{-1}$
with $cdc^{-1}d^{-1}WZYX$.

Repeat. ✓

Step 7 $M \cong (\mathbb{T}^2)^{\# n}$ or $(\mathbb{RP}^2)^{\# n}$ for some $n \geq 1$.

Note $M \cong S^2$ case covered in Step 2.

- At this point, M has a presentation which has all twisted pairs adjacent, all complementary pairs w/ no other edges intervening, $(aba^{-1}b^{-1})$. If all complementary, get $(\mathbb{T}^2)^{\# n}$; if all twisted get $(\mathbb{RP}^2)^{\# n}$; if mixed, appeal to the lemmata and get $(\mathbb{RP}^2)^{\# n}$.

$\square$

Euler characteristic

For X a finite CW cpx of dimn n, write $n_k :=$ # k-cells of X. Define the Euler characteristic of X,

$$\chi(X) := \sum_{k=0}^{n} (-1)^k n_k .$$

When we study homology, we will prove that $X \cong Y \Rightarrow \chi(X) = \chi(Y)$

Until then, we can get by with:

Prop The Euler characteristic of a polygonal presentation is unchanged by elementary transformations.

Pf Check the rules one by one and observe that any in/decreases in cells are compensated for by de/increases in cell of equal magnitude. $\square$

Prop $\chi(S^2) = 2$

$\chi((\mathbb{T}^2)^{\#n}) = 2-2n$

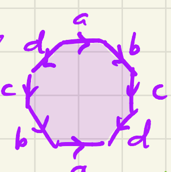
$\chi((\mathbb{RP}^2)^{\#n}) = 2-n$. $\square$

"chi"

k	n_k
0	1
1	4
2	1

$$\chi = 1-4+1 = -2$$

$\Rightarrow (\mathbb{T}^2)^{\#2}$ or $(\mathbb{RP}^2)^{\#4}$??

TPS Compute $\chi\Bigg($  $\Bigg)$. Which surfaces might this be? $|\langle a,b,c,d \mid aabbccdd\rangle|$

(2) Still don't know $\mathbb{RP}^2 \not\cong \mathbb{T}^2$.

Orientability

- Call a surface presentation $\mathcal{P}$ oriented if it has no twisted edge pairs. Then coloring tops red, bottom sides blue of polygons, we see that $|\mathcal{P}|$ is consistently colored.
- Call a compact surface orientable if it admits an oriented presentation.

⚠ This is a specialized, ad hoc version of orientability.

Prop A compact surface is orientable iff $\cong S^2$ or $(T^2)^{\#n}$.

Pf Check that no reflections are necessary in the algorithm from the classification thm when the original presentation is orientable. Thus we necessarily reduce to a standard presentation of S^2 or $(T^2)^{\#n}$. $\square$