28. X. 22

- Call a polygonal presentation a surface presentation if each symbol in S appears exactly twice in $W_1, \dots, W_k$. By the prop, $|\mathcal{P}|$ is a compact surface in this case.

- If $X \cong |\mathcal{P}|$, call $\mathcal{P}$ a presentation of X.

- If $|\mathcal{P}_1| \cong |\mathcal{P}_2|$, write $\mathcal{P}_1 \cong \mathcal{P}_2$ and call $\mathcal{P}_1, \mathcal{P}_2$ topologically equivalent.

Prop The following elementary transformations of polygonal presentations produce topologically equivalent presentations: (convention: $e \notin S$)

- Relabeling: e.g. $\langle a, b \mid aba^{-1}b^{-1} \rangle \mapsto \langle b, c \mid bcb^{-1}c^{-1} \rangle$

inverses
- Subdividing: replace every a with ae, a^{-1} with $e^{-1}a^{-1}$ $\quad \cancel{a} \mapsto \cancel{e}^{\cancel{a}}$
- Consolidating: if a, b always appear as ab or $b^{-1}a^{-1}$, replace each ab with a, $b^{-1}a^{-1}$ with a^{-1}.

- Reflecting: $\langle S \mid a_1 \cdots a_m, W_2, \dots, W_k \rangle \longmapsto \langle S \mid a_m^{-1} \cdots a_1^{-1}, W_2, \dots, W_k \rangle$
- Rotating: $\langle S \mid a_1 a_2 \cdots a_m, W_2, \dots, W_k \rangle \longmapsto \langle S \mid a_2 \cdots a_m a_1, W_2, \dots, W_k \rangle$

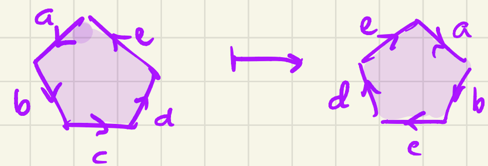
inverses
- Cutting: $\langle S \mid W_1 W_2, W_3, \dots, W_k \rangle \longmapsto \langle S, e \mid W_1 e, e^{-1} W_2, W_3, \dots, W_k \rangle$
- Pasting: reverse cutting

$(e^{-1})^{-1} = e$

inverses
- Folding: $\langle S, e \mid W_1 e e^{-1}, W_2, \dots, W_k \rangle \longmapsto \langle S \mid W_1, W_2, \dots, W_k \rangle$
- Unfolding: reverse folding

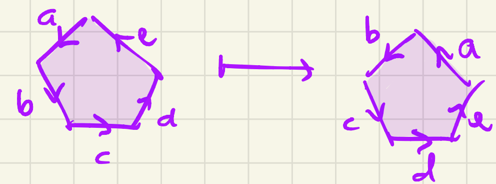
Pf Subdividing: Glue a e to a e instead of a to a. ✓

Reflecting:



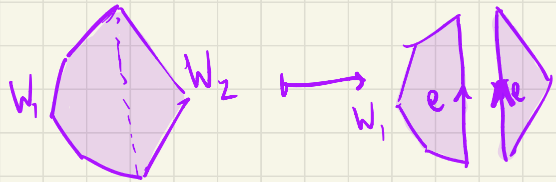
✓

Rotating:

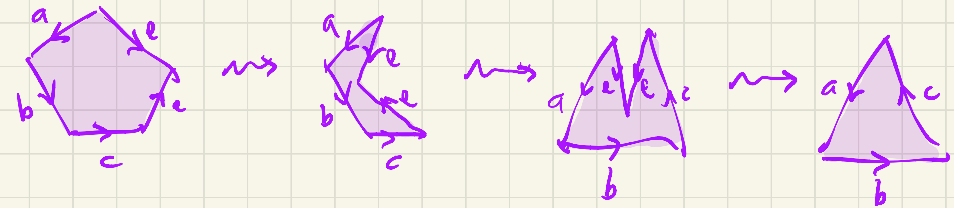


✓

Cutting:



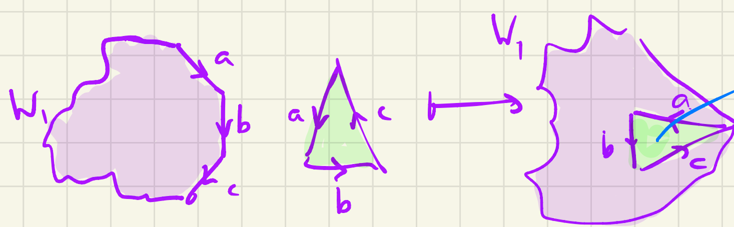
Folding:



$\square$

Prop Let M_1, M_2 be surfaces admitting presentations $\langle S_1 \mid W_1 \rangle$, $\langle S_2 \mid W_2 \rangle$ resp., with $S_1 \cap S_2 = \varnothing$. Then $|\langle S_1, S_2 \mid W_1 W_2 \rangle| \cong M_1 \# M_2$.

Pf $\langle S_1, a, b, c \mid W_1 c^{-1} b^{-1} a^{-1}, abc \rangle \cong \langle S_1 \mid W_1 \rangle$ via paste, fold, fold:



B_1 — regular coordinate disk in M_1.

(see p. 172)

Then $|\langle S_1, a, b, c \mid W_1 c^{-1}b^{-1}a^{-1} \rangle| \cong M_1 \setminus B_1$

Similarly, $|\langle S_2, a, b, c \mid abcW_2 \rangle| \cong M_2 \setminus B_2$

reg coord ball

Thus $|\langle S_1, S_2, a, b, c \mid W_1 c^{-1}b^{-1}a^{-1}, abcW_2 \rangle| \cong M_1 \# M_2$

By paste, fold, fold, this presentation is $\cong \langle S_1, S_2 \mid W_1 W_2 \rangle$. $\square$

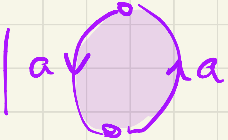
E.g. • $|\langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle| \cong |\langle a, b \mid aba^{-1}b^{-1} \rangle| \# |\langle c, d \mid cdc^{-1}d^{-1} \rangle|$

$\cong \mathbb{T}^2 \# \mathbb{T}^2$ (see 21.X.22 lecture).

• More generally, $(\mathbb{T}^2)^{\# n} \cong |\langle a_1, b_1, \ldots, a_n, b_n \mid [a_1, b_1] \cdots [a_n, b_n] \rangle|$

for $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ the "commutator" of a_i, b_i.

"standard presentations"

- $(\mathbb{RP}^2)^{\#n} \cong \left|\langle a_1,\dots,a_n \mid a_1 a_1 \cdots a_n a_n \rangle\right|$ $\qquad$ $\Big|$ 

$\Big| \cong \mathbb{RP}^2$

Classification

Thm Every compact surface admits a polygonal presentation.

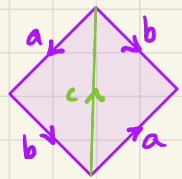
This follows from the triangulability of compact 2-mflds, a hard thm. $\square$

Thm (Classification of compact surfaces, Part I) Every nonempty compact connected 2-mfld is homeomorphic to one of the following:
(a) S^2, (b) $(\mathbb{T}^2)^{\#n}$ for some $n \geq 1$, (c) $(\mathbb{RP}^2)^{\#n}$ for some $n \geq 1$.

Presently, we can't tell whether some of the surfaces in this list might coincide (up to homeo). We'll need π_1 & the Seifert van Kampen theorem to prove they are in fact distinct!

Lemma The Klein bottle $K = |\langle a, b \mid abab^{-1} \rangle| \cong \mathbb{RP}^2 \# \mathbb{RP}^2$.

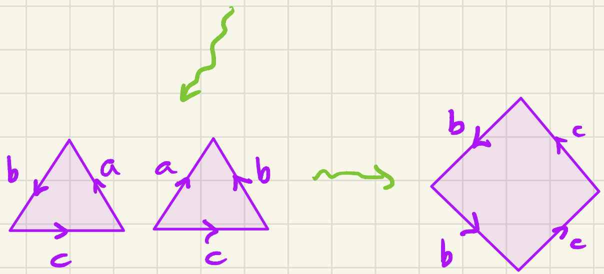
Pf



$$\langle a, b \mid abab^{-1} \rangle \cong \langle a, b, c \mid abc, c^{-1}ab^{-1} \rangle \quad \text{(cut)}$$

$$\cong \langle a, b, c \mid bca, a^{-1}cb \rangle \quad \text{(rotate, reflect)}$$

$$\cong \underbrace{\langle b, c \mid bbcc \rangle}_{\text{standard presentation of } \mathbb{RP}^2 \# \mathbb{RP}^2} \quad \text{(paste, rotate)}$$

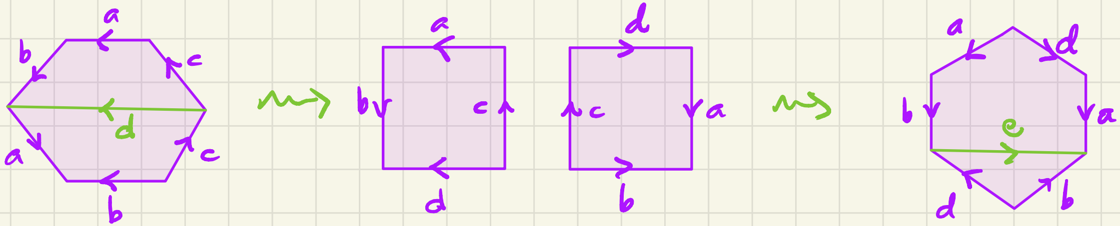


$\square$

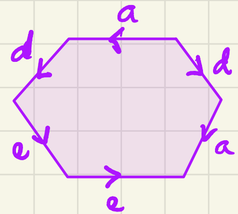
Lemma $\mathbb{T}^2 \# \mathbb{RP}^2 \cong (\mathbb{RP}^2)^{\#3}$

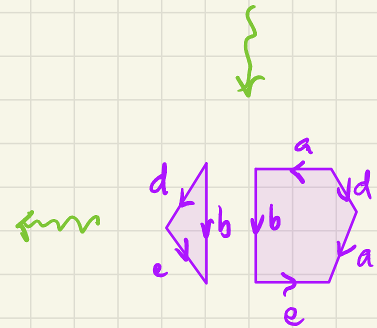
Pf First note $P := \langle a, b, c \mid abab^{-1}cc \rangle$ is a presentation of $K \# \mathbb{RP}^2$.

By the previous lemma, $|P| \cong (\mathbb{RP}^2)^{\#3}$. Now show $|P| \cong \mathbb{T}^2 \# \mathbb{RP}^2$:



$\square$ $\mathbb{T}^2 \# \mathbb{RP}^2 \cong$





Upshot If we have an $\mathbb{RP}^2$ in a connect sum decomposition w/ $\mathbb{RP}^2$, $\mathbb{T}^2$, K then every summand becomes $\mathbb{RP}^2$!

Note $M \# S^2 \cong M$.

Q What is the monoid of compact surfaces up to $\cong$ under $\#$? (Assuming Part II.)

Pf of Classification I Assume M is a connected compact mfld equipped w/ a presentation $\mathcal{P}$ (by the lemma). Call a pair of edges complementary if labeled a, a^{-1}; twisted if a, a.

Note Conditions in steps are cumulative.

Step 1 M admits a presentation with exactly one face:

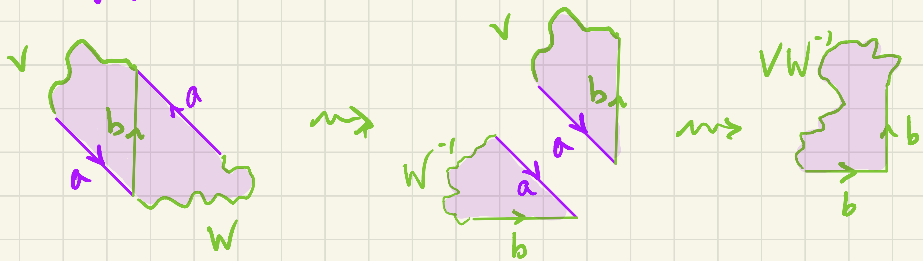
- For induction, assume true when $\mathcal{P}$ has n faces for some $n \geq 1$. If $\mathcal{P}$ has $n+1$ faces, connectedness of M implies $(n+1)$-th face shares an edge with one of the other faces. Paste to get a presentation w/ n faces, then use the induction hypothesis. ✓

Step 2 Either (a) $M \cong S^2$ or (b) admits a presentation with no adjacent complementary pairs:

- Eliminate adjacent pairs by folding. This terminates in (b) or $\langle a \mid aa^{-1} \rangle$ which realizes to S^2. ✓

Step 3 M admits a presentation in which all twisted pairs are adjacent:

- If a twisted pair a, a is not adjacent, rotate to $VaWa$ with V, W nonempty words. Transform via



into $VW^{-1}bb$. This decreases nonadjacent pairs (twisted and complementary) by at least one, so after finitely many steps all pairs are adjacent. Use Step 2 to eliminate adjacent complementary pairs. ✓