26. X. 22

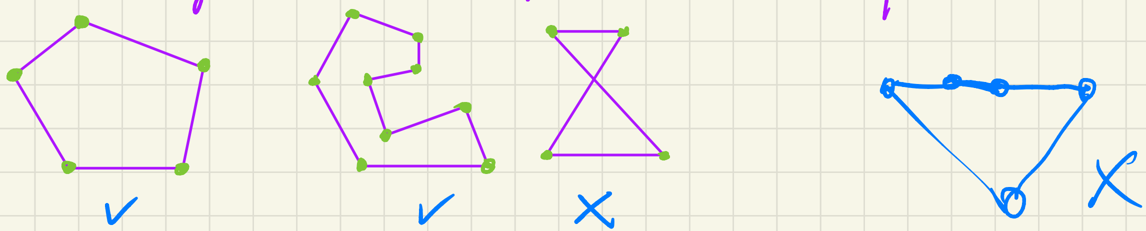
Skipping simplicial complexes!

- They're great, but they won't appear later in the course; you're not responsible for the content on pp. 147-155.

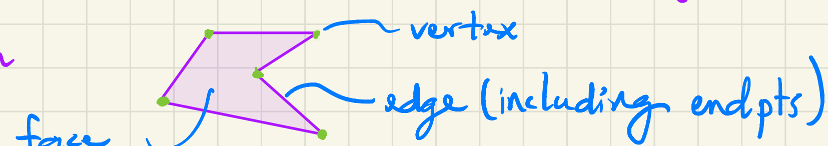
Surfaces

Surface := 2-dimensional mfld (w/o ∂)

Polygon := union of finitely many closed line segments in $\mathbb{R}^2$ that meet only at their endpoints, homeomorphic to S^1



Polygonal region := compact subset of $\mathbb{R}^2$ w/ interior a regular coordinate ball, ∂ a polygon

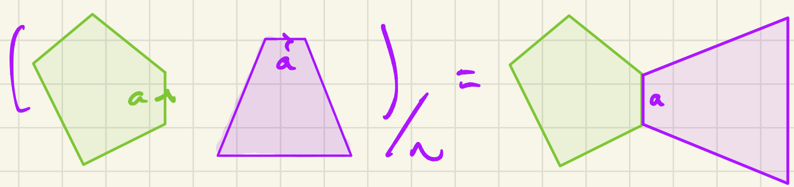


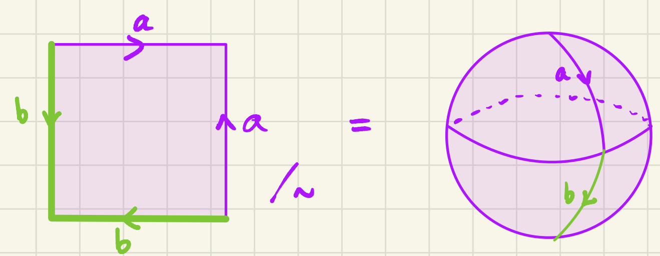
Prop Let $P_1, \dots, P_k$ be polygonal regions, $P = P_1 \sqcup \cdots \sqcup P_k$, $\sim$ an equivalence rel'n on P identifying some edges with others via affine homeomorphisms (i.e. translating, rotating, reflecting, scaling).

(a) $P/\sim$ is a finite 2-dim'l CW cpx w/ 0-skeleton = images of vxs, 1-skeleton = image of edges.

(b) If $\sim$ identifies each edge in P w/ exactly one other edge, then $P/\sim$ is a compact surface.

E.g.





Pf Let $\pi : P \longrightarrow M := P/\sim$ be the quotient map,

$$M_0 := \pi\{vxs\}, \quad M_1 := \pi\,\partial P, \quad M_2 := M.$$

Then M_0 is discrete, and for $k=1,2$, $M_k = M_{k-1} \cup_\varphi$ (finitely many k-cells).

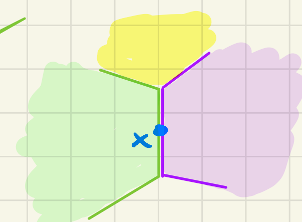
Thus M is a finite 2-dim'l CW cpx by defn.

Euclideanity?

For (b), we now only need to check local Euclideanness.

If $x = \pi$(interior pt of P) ✓

If $x = \pi$(edge $\setminus$ vxs pt of P):

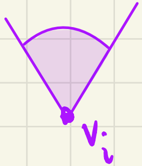


✓

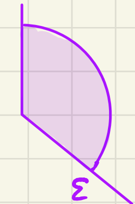
If $x = \pi$(vx of P), there is something to check:

$\pi^{-1}\{x\} = \{v_1, \dots, v_\ell\} \subseteq \{$vxs of $P\}$. Choose small $\varepsilon > 0$ s.t.

$B_\varepsilon(v_i) \cap \{vxs\} = \{v_i\}$, $B_\varepsilon(v_i) \cap \{edges\}$ contains no edges other than those incident with v_i. Then $B_\varepsilon(v_i) \cap P_j$ is of the form

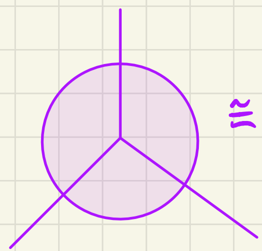
 and we may form a homeo

$$B_\varepsilon(v_i)\cap P_j \cong \left\{\exp(ir\theta) \;\middle|\; \begin{array}{l}\theta_0\le\theta\le\theta_0+2\pi/\ell \\ 0\le r<\varepsilon\end{array}\right\} =$$



(here $\ell=3$)

Glue these together to get



$\cong B_\varepsilon(0)$ ✓ $\square$

<u>Building surfaces</u> Recall from HW, $M_1 \# M_2$ the connected sum of M_1, M_2.

In general, there are <u>two</u> connected sums depending on whether $\partial B_1 \xrightarrow{\cong} \partial B_2$ is orientation-preserving or reversing.

Later we'll be able to prove connected sums of compact surfaces are unique up to homeo.

<u>E.g.</u> $\mathbb{T}^2 \# M \cong (M \text{ with a handle attached}) := M_0 \cup_\varphi (S^1 \times [0,1])$

for $M_0 = M \setminus (2 \text{ reg. coord balls})$, $\varphi : S^1 \times \{0,1\} \to M_0$ attaching the ends of $S^1 \times [0,1]$ to M_0:

Why?

<u>Sub-e.g.</u> $(\mathbb{T}^2)^{\# n} \cong (S^2 \text{ w/ } n \text{ handles attached})$

Polygonal presentations of surfaces

Given a set S, a word in S is an ordered k-tuple (written as a string) of symbols of the form a or a^{-1} for $a \in S$.

A polygonal presentation $\mathcal{P} = \langle S \mid W_1, \dots, W_k \rangle$ is a finite set S together with finite words $W_1, \dots, W_k$ of length ≥ 3 s.t. every symbol of S appears in at least one word.

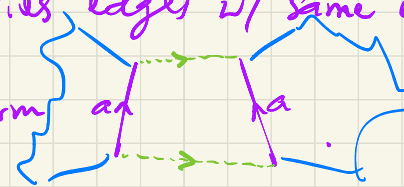
Convention: $\langle \{a,b\} \mid aba^{-1}b^{-1} \rangle =: \langle a,b \mid aba^{-1}b^{-1} \rangle$.

Also allow $\langle a \mid aa \rangle$, $\langle a \mid a^{-1}a^{-1} \rangle$, $\langle a \mid aa^{-1} \rangle$, $\langle a \mid a^{-1}a \rangle$.

The geometric realization of $\mathcal{P}$, $|\mathcal{P}|$, is the following space:

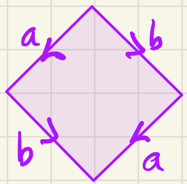
(1) For each W_i, let P_i be the unit regular convex k-gon w/ vx on positive y-axis where $k = \text{length}(W_i)$.

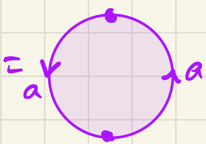
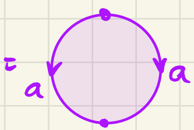
(2) Start at top point of P_i & label edges counterclockwise w/ symbols of W_i.
(a — label w/ a in ccw direction. a^{-1} — label w/ a in cw direction)

(3) Define $\sim$ on $\coprod_{i=1}^{l} P_i$ which identifies edges w/ same edge symbol via an affine homeo of the form 

(4) Set $|\mathcal{P}| = \coprod_{i=1}^{l} P_i \big/ \sim$.

Note: Use a bi-gon for presentations $\langle a \mid aa \rangle$, $\langle a \mid aa^{-1} \rangle$ etc.

E.g. $|\langle a, b \mid aba^{-1}b^{-1} \rangle| =$  $/\sim \;\cong T^2$,

$|\langle a \mid aa \rangle| =$  $\cong \mathbb{R}P^2$, $\quad |\langle a \mid aa^{-1} \rangle| =$  $\cong S^2$.

TPS Determine the homeomorphism types of $|\langle a, b \mid abb^{-1}a \rangle|$ & $|\langle a, b \mid abab^{-1} \rangle|$.