24. X .22

Hopf map and $\mathbb{C}P^2$

- $$\begin{tikzcd} S^3 \arrow[d, phantom, "\cap"] \arrow[rr, bend left, "\eta"] & & \\ \mathbb{C}^2 \setminus \{0\} \arrow[r] & \mathbb{C}^2\setminus\{0\} / \mathbb{C}^\times =: \mathbb{C}P^1 \cong S^2 \end{tikzcd}$$

  $U_0 = \{[z:1] \mid z \in \mathbb{C}\} \cong \mathbb{C}$

  $\mathbb{C}P^1 \setminus U_0 = \{[1:0]\} \Rightarrow \mathbb{C}P^1 =$ one-point compactification of $\mathbb{C}$

- $$\begin{tikzcd} S^3 \arrow[r, "\eta"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner" very near start] & S^2 \arrow[d] \\ e^4 \arrow[r] & S^2 \cup_\eta e^4 \cong e^0 \cup e^2 \cup_\eta e^4 \end{tikzcd}$$

- $\mathbb{C}P^2 := \mathbb{C}^3\setminus\{0\} / \mathbb{C}^\times \supseteq \mathbb{C}P^1$ as points $\{[x:y:0] \mid (x,y) \in \mathbb{C}^2 \setminus \{0\}\}$

  $\mathbb{C}P^2 \setminus \mathbb{C}P^1 = \{[x:y:1] \mid (x,y) \in \mathbb{C}^2\} \cong \mathbb{C}^2 \cong \mathbb{R}^4 \cong (e^4)^\circ$

Challenge Prove $\mathbb{C}P^2 \cong e^0 \cup e^2 \cup_\eta e^4$ and that $\mathbb{C}P^n$ has a CW structure with one cell in each even dimn $2k$, $0 \le 2k \le 2n$.

Products of CW complexes

If X, Y are CW complexes with characteristic maps $\Phi_\alpha : e^m_\alpha \to X$, $\Phi_\beta : e^n_\beta \to Y$, then $e^m_\alpha \times e^n_\beta$ is an $(m+n)$-cell and we have a map

$$\Phi_\alpha \times \Phi_\beta : e^m_\alpha \times e^n_\beta \to X \times Y.$$
$$(a,b) \mapsto (\Phi_\alpha(a), \Phi_\beta(b))$$

These are not necessarily the characteristic maps of a CW structure on $X \times Y$!

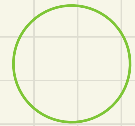
Given a space X, write X_c for the compactly gen'd topology on X with $U \subseteq X_c$ open iff $U \cap K \subseteq K$ open $\forall K \subseteq X$ compact. (Potentially finer.)

Thm $\Phi_\alpha \times \Phi_\beta$ are characteristic maps for a CW structure on $(X \times Y)_c$.

If either X or Y is (locally) compact, then $(X \times Y)_c = X \times Y$; also true when both X, Y have countably many cells.

Classification of 1-mflds

THM Every compact connected 1-mfld is $\cong S^1$; every non-compact connected 1-mfld is $\cong \mathbb{R}$.



or

We'll prover this via regular CW decompositions.

Thm Every 1-mfld admits a regular CW decomposition.

Pf Let M be a 1-mfld. Then it has a countable cover $\{U_i\}_{i\in\mathbb{N}}$ by regular coordinate balls. Here $U_i \cong \mathbb{B}^1 = (0,1)$, $\overline{U}_i \cong \overline{\mathbb{B}}^1 = [0,1]$.

Let $M_n = \overline{U}_0 \cup \dots \cup \overline{U}_n \Rightarrow M = \bigcup_{n\in\mathbb{N}} M_n$.

Construct a finite regular cell decomp'n $\mathcal{E}_n$ of M_n s.t. M_{n-1} is a subcomplex of M_n.

Once we do this: $\mathcal{E} = \bigcup \mathcal{E}_n$ has pairwise disjoint cells with union M.

For any $x \in M$, $\exists n$ s.t. $x \in U_n \subseteq M_n$, whence cells of $\mathcal{E} \setminus \mathcal{E}_n$ are disjoint from $M_n \Rightarrow U_n$ a nbhd of x intersecting no cells of $\mathcal{E}$ except those in $\mathcal{E}_n$.

Thus $\mathcal{E}$ is locally finite $\Rightarrow$ CW. ✓

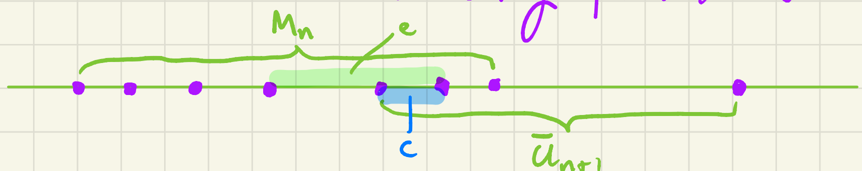
It remains to construct $\mathcal{E}_n$: let $\mathcal{E}_0 = \{$1-cell U_0, two ∂ 0-cells of $\overline{U}_0\}$

This is a regular cell decomp'n of $M_0 = \overline{U}_0$.

Fix $n \geq 0$ and assume (for strong induction) that for $i = 0, \dots, n$ we've built $\mathcal{E}_i$.

Construct a finite regular cell decomp $\mathcal{C}$ of $\overline{U}_{n+1} \cong [0,1]$ by taking 0-cells = 0-cells of $\mathcal{E}_n$ in U_{n+1} together with $\partial \overline{U}_{n+1}$;

1-cells = intervening open intervals.



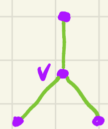
<u>Claim</u> Each cell in $\mathcal{C}$ is either contained in a cell of $\mathcal{E}_n$ or is disjoint from all cells of $\mathcal{E}_n$.

Indeed, if e is a 1-cell in $\mathcal{E}_n$ s.t. $c \cap e \neq \emptyset$ for some 1-cell $c \in \mathcal{C}$, then $c \cap \bar{e} = c \cap e$. Since $e \subseteq M$ open, $\bar{e} \in M$ closed, get $c \cap e$ clopen in c. Since c connected, $c \cap e = c \Rightarrow c \subseteq e$. ✓

Take $\mathcal{E}_{n+1}$ = union of $\mathcal{E}_n$ w/ cells in $\mathcal{C}$ not contained in any cells of $\mathcal{E}_n$. This works! □

<u>Lemma</u> For M a 1-mfld w/ regular CW decomp'n, the ∂ of every 1-cell has exactly two 0-cells, and every 0-cell of M bounds exactly two 1-cells.

<u>Pf</u> Read 5.2G.



$\setminus \{v\}$ has >2 conn'd components, but v has a nbhd $\cong \mathbb{R}^1$ ⚡

<u>Pf of THM</u> Endow 1-mfld M with a 1-dim regular CW decomp. By lemma, M is a graph in which each edge has 2 vxs and every vx has deg 2. Thus we can (inductively) build vx, edge bi-infinite sequences

$(v_i)_{i\in\mathbb{Z}}$, $(e_j)_{j\in\mathbb{Z}}$ s.t. $\underset{v_{j-1}}{\bullet}\overset{e_j}{\frac{\qquad}{}}\underset{v_j}{\bullet}\overset{e_{j+1}}{\frac{\qquad}{}}\bullet$ $\forall j\in\mathbb{Z}$.

For $n\in\mathbb{Z}$, let
$$F_n : [n-1,n] \xrightarrow{\cong} \bar{e}_n$$
$$n-1 \longmapsto v_{n-1}$$
$$n \longmapsto v_n$$
and glue to get $F:\mathbb{R}\to M$.

Case 1 All vxs v_n are distinct. Then $\bar{e}_m \cap \bar{e}_n \neq \emptyset$ iff $m=n-1,n,n+1$

$\Rightarrow F$ injective. $B\subseteq M$ compact $\Rightarrow B\subseteq$ finite subcomplex $\Rightarrow F^{-1}B\subseteq[-c,c]$

so compact. Thus F is proper, so im F is closed. Since M conn'd,

suffices to show im F open. Have $Y_n = \overset{e_n}{\frac{\qquad}{}}\underset{v_n}{\circ}\overset{e_{n+1}}{\frac{\qquad}{}}$ open

and $F((n-1,n+1)) = Y_n$, so im $F = \bigcup_n Y_n$ so $M\cong\mathbb{R}$.

Case 2 $v_j = v_{j+k}$ (chosen so that k is the smallest such positive integer).

Set $\hat{F} = F|_{[j,j+k]}$. Check $\hat{F}$ is a quotient map.

But the only identification is $\hat{F}(j) = \hat{F}(j+k)$.

This is the same identification as the quotient map

$$G : [j, j+k] \longrightarrow S^1, \quad t \longmapsto \exp(2\pi i t/k)$$

, so by uniqueness of quotients,

$M \cong S^1$. $\square$

Cor A conn'd 1-mfld w/ $\partial \neq \emptyset$ is $\cong [0,1]$ if compact, $[0,\infty)$ if not.

Pf Let M be such a mfld and $D(M)$ its double:

$$\begin{tikzcd} \partial M \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & M \arrow[d] \\ M \arrow[r] & D(M) \end{tikzcd}$$

(E.g., $D[0,1] \cong S^1$, $D[0,\infty) \cong \mathbb{R}$.)

Since $D(M)$ is a 1-mfld, it is $\cong S^1$ or $\mathbb{R}$, and M is $\cong$ proper conn'd subspace of $D(M)$. If $D(M) \cong S^1$, choose $p \in D(M) \setminus M$ to get $M \hookrightarrow D(M) \setminus \{p\} \cong \mathbb{R}$, so in both cases M is $\cong$ conn'd subset of $\mathbb{R}$ containing >1 pt, which is thus an interval.

Closed bdd interval $\Rightarrow M \cong [0,1]$.

O/w $[a,b)$, $(a,b]$, $[a,\infty)$, $(-\infty,a]$ — all $\cong [0,\infty)$. $\square$