21.X.22

Cell & CW complexes

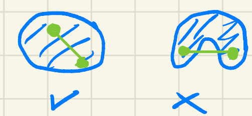
A <u>open n-cell</u> is a space $e^\circ \cong \mathbb{B}^n$.  $e^\circ \cong \mathbb{B}^2$

A <u>closed n-cell</u> is a space $e \cong \bar{\mathbb{B}}^n$  $e \cong \bar{\mathbb{B}}^2$

② Lee writes e and $\bar{e}$.

Fact (Prop 5.1) Any compact $\underbrace{\text{convex}}$ subspace of $\mathbb{R}^n$ is a closed n-cell.

line segment joining any two pts in the set is contained in the set.

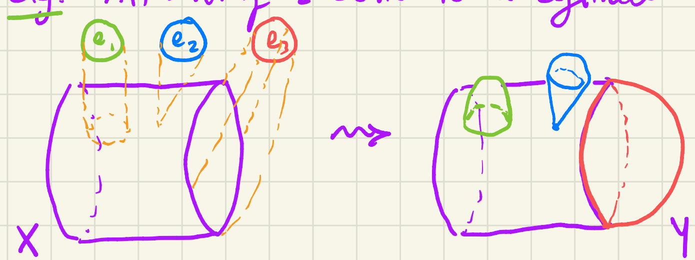


E.g. A solid icosahedron is a closed 3-cell.

Say that Y is constructed from X by <u>attaching n-cells</u> when it is of the form

$$\begin{array}{ccc} \coprod_{\alpha \in A} \partial e^n_\alpha & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow \\ \coprod_{\alpha \in A} e^n_\alpha & \longrightarrow & Y \end{array}$$

E.g. Attaching 2-cells to a cylinder:



Given a family of subspaces $\mathcal{B}$ with union X, call the topology of X coherent with $\mathcal{B}$ when $U \subseteq X$ open $\iff$ $U \cap B \subseteq B$ open $\forall B \in \mathcal{B}$.

E.g. Compactly generated spaces X are coherent with $\{K \subseteq X \mid K \text{ compact}\}$.

Defn A cell complex is a H'ff space X and subspaces $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X$ such that (1) X_0 is discrete

(2) X_n is formed from X_{n-1} by attaching n-cells:

$$\begin{tikzcd} \coprod_{\alpha \in A} \partial e^n_\alpha \arrow[r, "\varphi"] \arrow[d] & X_{n-1} \arrow[d] \\ \coprod_{\alpha \in A} e^n_\alpha \arrow[r] & X_n = X_{n-1} \cup_\varphi \coprod_{\alpha \in A} e^n_\alpha \end{tikzcd}$$

"attaching map"

(Note: $A = \varnothing$ allowed!)

$$\begin{tikzcd} e^n_\alpha \arrow[r] \arrow[dr, "\Phi_\alpha"'] & X_n \arrow[d, hook] \\ & X \end{tikzcd}$$

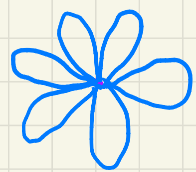
"characteristic map" of e^n_α

Note: $\Phi_\alpha|_{(e^n_\alpha)^\circ}$ embedding

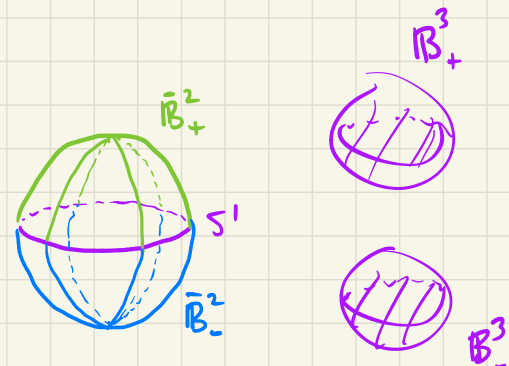
Call X_n the n-skeleton of X. If $X = X_n$ for some n, X is finite dimensional and the smallest n such that $X = X_n$ is the dimension of X.

E.g. • A 1-dimensional cell complex is a graph

• A 1-dim'l cell complex with $X_0 = *$ is a bouquet of circles $\bigvee S^1$



•
$$\begin{tikzcd}
S^{n-1}_+ \amalg S^{n-1}_- \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & S^{n-1} \arrow[d] \\
\bar{B}^n_+ \amalg \bar{B}^n_- \arrow[r] & S^n
\end{tikzcd}$$



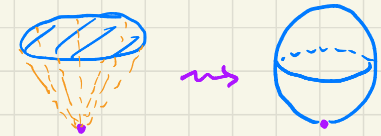
gives a presentation of S^n as an n-dimensional cell complex ~~with 2 n-cells and 1 (n-1)-cell.~~ inductively.

The cells are regular because their characteristic maps are embeddings.

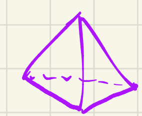
- We can also build S^n from one 0-cell + one n-cell:

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & * \arrow[d] \\ \bar{\mathbb{B}}^n \arrow[r] & \bar{\mathbb{B}}^n / S^{n-1} \cong S^n \end{tikzcd}$$

not regular



- Any convex polyhedron presents S^2 as a cell complex with #faces 2-cells, #edges 1-cells, #vxs 0-cells.



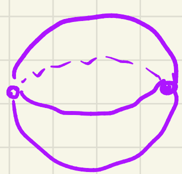
Defn A cell complex is a CW complex when
(C) the closure of each open cell is contained in a union of finitely many cells.
(W) the topology is coherent with the family of closed cells.

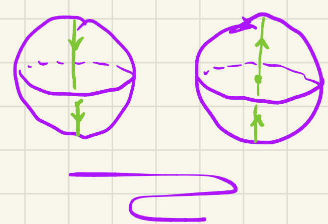
C = closure finiteness, W = "weak topology"

Above examples are CW, and C is automatic in our presentation.
If X is finite-dimensional, W is automatic; if it's infinite dimensional, it admits a unique W topology, possibly finer than its given topology.

(2) Text defines *cell decompositions* of X as a partition $\mathcal{E}$ of a space X into open cell subspaces e° admitting characteristic maps $\bar{\mathbb{B}}^n \xrightarrow{\Phi} X$ s.t. $\Phi|_{\mathbb{B}^n}$ is a homeo onto e° and maps $\partial \bar{\mathbb{B}}^n = S^{n-1}$ into the union of cells of lower dimn. Props 5.18 + 5.20 show this is equivalent.

E.g. The infinite dimensional sphere: Recall S^n presented as two n-cells attached to S^{n-1}. Don't stop $\rightsquigarrow S^\infty$.





Loop in S^3

CW complexes are nice

J.H.C. Whitehead (1904-1960) and his pig

5.11 path conn'd $\Leftrightarrow$ conn'd $\Leftrightarrow$ 1-skeleton conn'd $\Leftrightarrow$ some n-skeleton conn'd

finitely many cells

5.12 closure of each cell $\subseteq$ finite subcomplex

union of cells containing the closure of each cell

5.13 $A \subseteq X$ discrete $\Leftrightarrow$ $A \cap e$ finite for all cells e

5.14 $A \subseteq X$ compact $\Leftrightarrow$ A closed & $\subseteq$ finite subcomplex

5.15 X compact $\Leftrightarrow$ finite

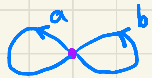
5.22 paracompact

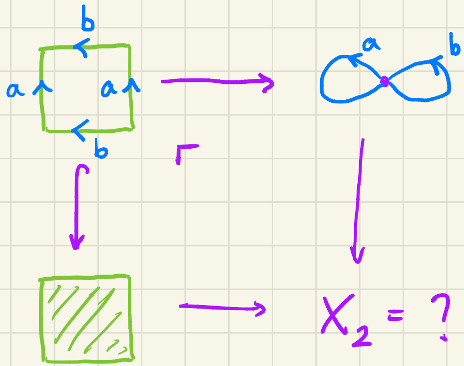
5.23 countably many cells + locally Euclidean $\Rightarrow$ manifold

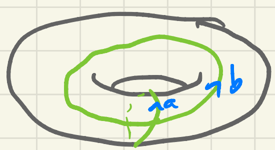
5.24 for CW mflds, CW dimn = mfld dimn.

TPS

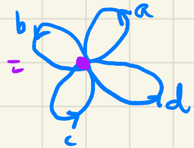
$X_0 = *$

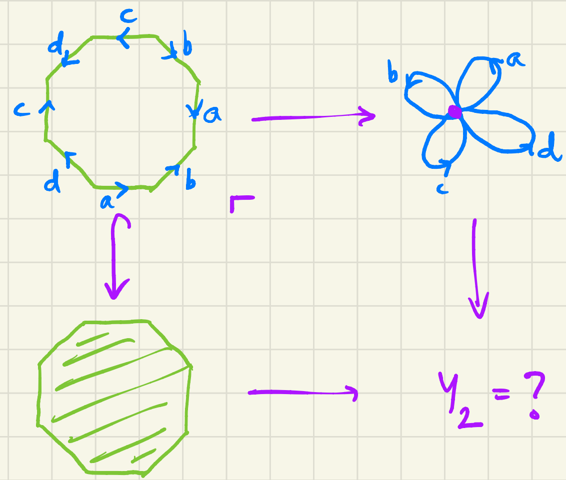
$X_1 =$ 

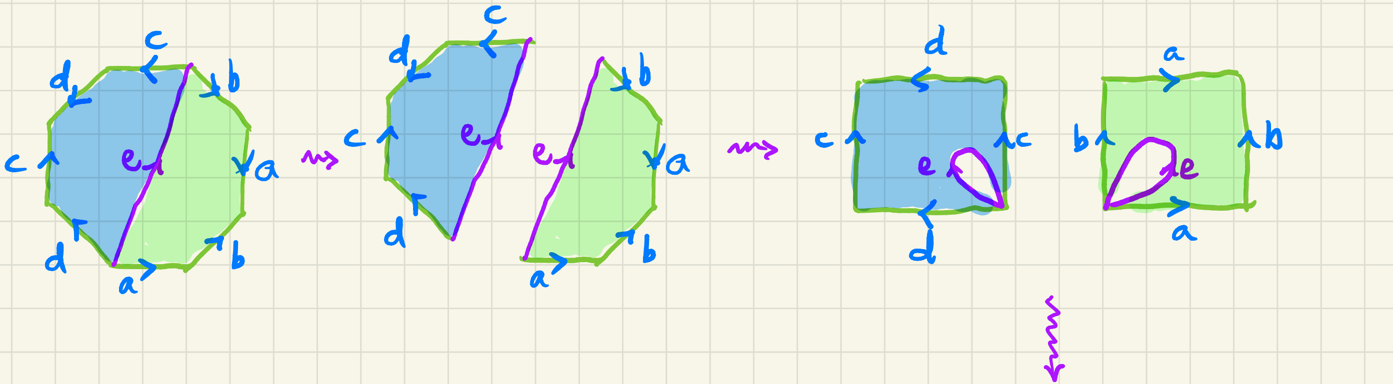


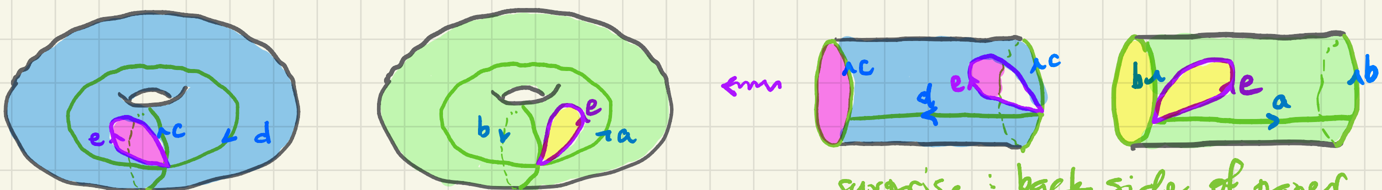


$Y_0 = *$

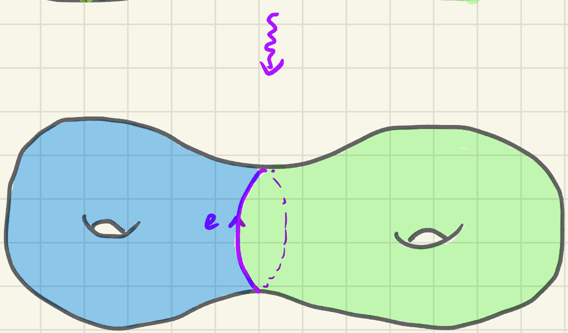
$Y_1 =$ 







surprise: back side of paper
is a different color!



<u>Exercise</u> Draw a, b, c, d on
the diagram