19.X.22

Partitions of unity

If $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ is an indexed open cover of X, a partition of unity subordinate to $\mathcal{U}$ is a family of cts functions $\psi_\alpha : X \to \mathbb{R}$, $\alpha \in A$ s.t.

(i) $0 \leq \psi_\alpha(p) \leq 1 \quad \forall \alpha \in A, p \in X$

(ii) $\operatorname{supp} \psi_\alpha \subseteq U_\alpha$

(iii) $(\operatorname{supp} \psi_\alpha)_{\alpha \in A}$ is locally finite

(iv) $\displaystyle\sum_{\alpha \in A} \psi_\alpha(p) = 1$ for all $p \in X$.

Note By (iii), the sum in (iv) is finite for each $p \in X$.

Lemma X paracompact H'ff, $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ indexed open cover of X then $\mathcal{U}$ admits a locally finite open refinement $\mathcal{V} = (V_\alpha)_{\alpha \in A}$ indexed by the same set s.t. $\overline{V_\alpha} \subseteq U_\alpha \quad \forall \alpha \in A$.

$\therefore$ {strengthened paracompactness condition

Pf Read 4.84. $\square$

$(U_\alpha)_{\alpha\in A}$ "

Thm (existence of partitions of unity) Let X be paracompact H'ff. If $\mathcal{U}$ is any indexed open cover of X, then $\exists$ partition of unity subordinate to $\mathcal{U}$.

Pf Apply the lemma twice to get locally finite open covers $\mathcal{V} = (V_\alpha)_{\alpha\in A}$, $\mathcal{W} = (W_\alpha)_{\alpha\in A}$ s.t. $\overline{W}_\alpha \subseteq V_\alpha$, $\overline{V}_\alpha \subseteq U_\alpha$. For each $\alpha\in A$, let $f_\alpha : X \to [0,1]$ be a bump function for $\overline{W}_\alpha$ supported in V_α. Define

$$f : X \longrightarrow \mathbb{R}$$
$$p \longmapsto \sum_{\alpha\in A} f_\alpha(p).$$

Since $\operatorname{supp} f_\alpha \subseteq V_\alpha$, $(\operatorname{supp} f_\alpha)_{\alpha\in A}$ is locally finite; thus each point of X has a nbhd on which only fin many f_α are nonzero, so f is cts. Since $\{W_\alpha\}_{\alpha\in A}$ covers X, f is positive everywhere. Thus we may define $\psi_\alpha(p) := f_\alpha(p) / f(p)$ to get the desired partition of unity. $\square$

<u>Thm</u> (embeddability of compact mflds) Every compact mfld is homeomorphic to a subset of some Euclidean space.

<u>Pf</u> Suppose M is a compact n-mfld covered by open $U_1, \dots, U_k \cong \mathbb{R}^n$; say $\varphi_i : U_i \xrightarrow{\cong} \mathbb{R}^n$. Let (ψ_i) be a partition of unity subordinate to this cover and define

$$F_i : M \longrightarrow \mathbb{R}^n$$
$$x \longmapsto \begin{cases} \psi_i(x)\varphi_i(x) & x \in U_i \\ 0 & x \in M \setminus \operatorname{supp} \psi_i, \end{cases}$$

cts by gluing lemma. Set

$$F : M \longrightarrow \mathbb{R}^{nk+k}$$
$$x \longmapsto (F_1(x), \dots, F_k(x), \psi_1(x), \dots, \psi_k(x))$$

<u>TPS</u> $nk+k$ for $M = \mathbb{RP}^2$ or S^2

F is cts, so by CML suffices to prove F is injective:

Suppose $F(x) = F(y)$. Since $\sum \psi_i(x) = 1$, $\exists i$ s.t. $\psi_i(x) > 0 \Rightarrow x \in U_i$.

Since $F(x) = F(y)$, $y \in U_i$ as well. Thus $F_i(x) = F_i(y)$, whence

$\varphi_i(x) = \varphi_i(y)$, so $x = y$ since $\varphi_i : U_i \cong \mathbb{R}^n$. $\square$

(Whitney: M embeds in $\mathbb{R}^{2n+1}$.)

Thm Suppose M is a mfld, $B \subseteq M$ closed. Then $\exists$ cts $f : M \to [0,\infty)$ s.t. $f^{-1}\{0\} = B$.

Pf (1) For $M = \mathbb{R}^n$, distance to B function $u(x) = \inf\{|x-y| \mid y \in B\}$ works.

(2) For gen'l M, let $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ be a cover of M by opens $\cong \mathbb{R}^n$, and let ψ_α be a subordinate partition of unity. By (1), have cts $u_\alpha : U_\alpha \to [0,\infty)$ with $u_\alpha^{-1}\{0\} = B \cap U_\alpha$. Define

$$f : M \longrightarrow \mathbb{R}$$
$$x \longmapsto \sum_{\alpha \in A} \underbrace{\psi_\alpha(x) u_\alpha(x)}_{0 \text{ outside supp } \psi_\alpha}.$$

This works. $\square$

Cor (manifolds are perfectly normal) M a mfld, $A, B \subseteq M$ disjoint closed. Then $\exists\, f: M \to [0,1]$ cts s.t. $f^{-1}\{1\} = A$, $f^{-1}\{0\} = B$.

Pf Have $u, v: M \to [0, \infty)$ with $u^{-1}\{0\} = A$, $v^{-1}\{0\} = B$. Then

$$f(x) = \frac{v(x)}{u(x) + v(x)}$$

works. $\square$

Proper maps ... { When you want to use CML but the domain isn't compact

- A function $F: X \to Y$ is **proper** when $\forall K \subseteq Y$ compact, $F^{-1}K \subseteq X$ is compact.

- A sequence (x_i) in X **diverges to ∞** when $\forall K \subseteq X$ compact almost every $x_i \notin K$. ... { (x_i) escapes every compact

Lemma Suppose X is first countable H'ff. A sequence in X diverges to ∞ iff it has no convergent subsequence.

Pf ($\Rightarrow$) Suppose (x_i) has a conv subseq $(x_{i_j}) \to x$. Then $K = \{x_{i_j} \mid j \in \mathbb{N}\} \cup \{x\}$ is compact and (x_i) doesn't escape it so (x_i) does not diverge to ∞.

($\Leftarrow$) Suppose (x_i) has no conv subseq. If $K \subseteq X$ compact contains ∞ly many x_i, then $\exists$ subseq (x_{i_j}) in K. But K is seq compact ⚔. $\square$

Prop Suppose $F: X \to Y$ is proper. Then F takes every seq diverging to ∞ in X to a sequence diverging to ∞ in Y.

Pf Suppose $(x_i) \to \infty$ in X and suppose for ⚔ that $(F(x_i)) \not\to \infty$ in Y. Then $\exists K \subseteq Y$ compact containing ∞ly many values $F(x_i)$, whence $F^{-1}K$ contains ∞ly many x_i. Since F is proper, $F^{-1}K$ is compact, ⚔. $\square$

Identifying proper maps If any of the following hold, then a cts map $F: X \to Y$ is proper:

(a) X compact, Y H'ff.

(b) X 2^{nd} countable H'ff, F takes all $(x_i) \to \infty$ to $(F(x_i)) \to \infty$

(c) F is closed with compact fibers

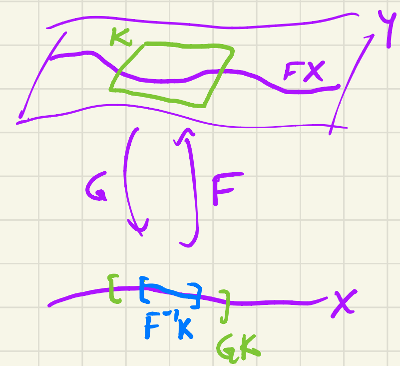
(d) F is an embedding with closed image

(e) Y is H'ff and F has cts left inverse

Add'lly, if F is proper and $A \subseteq X$ is saturated wrt F, then $F|_A : A \to FA$ is proper.

$A = F^{-1}B$ for some $B \subseteq Y$

Pf of (e) Y H'ff, $G: Y \to X$ cts s.t. $G \circ F = id_X$.



Take $K \subseteq Y$ compact. Since Y is H'ff, K is closed, so $F^{-1}K \subseteq X$ is closed.

But for $x \in F^{-1}K$, $G(F(x)) = x$

$\underset{GK}{\overset{\cap}{F(x)}}$

Thus $F^{-1}K \subseteq GK$ is closed $\subseteq$ compact $\Rightarrow F^{-1}K$ compact $\square$

Other proofs: read 4.93. $\square$

A space X is compactly generated when:

⊛ If $A \subseteq X$ s.t. $\forall K \subseteq X$ compact, $A \cap K$ closed, then A is closed.

Equiv: (open) (open)

Lemma First countable spaces and locally compact H'ff spaces are compactly gen'd.

Pf Space X, $A \subseteq X$ satisfying the hypothesis of ⊛. Suppose $x \in \overline{A}$. WTS $x \in A$.

(a) X first countable. Read :)

(b) X locally compact H'ff. Take $K \subseteq X$ compact containing a nbhd U of x.
If V is a nbhd of x, then $x \in \overline{A} \Rightarrow V \cap U$ contains a pt of A,
so V contains a pt of $A \cap K$. Thus $x \in \overline{A \cap K}$.

~~Since $A \cap K \subseteq K$ closed, $K \subseteq X$ closed (X H'ff),~~
~~get $A \cap K \subseteq X$ closed $\Rightarrow x \in A \cap K \subseteq A$.~~

By ⊛, $A \cap K$ closed $\Rightarrow x \in A \cap K \subseteq A$ □

Thm (proper cts maps are closed) Suppose X is any space, Y is a compactly gen'd H'ff space, and $F : X \to Y$ is a proper cts map. Then F is closed.

Pf Let $A \subseteq X$ be closed. We show FA closed by showing that $FA \cap K$ is closed $\forall K \subseteq Y$ compact. If $K \subseteq Y$ compact, then $F^{-1}K$ is compact, and $A \cap F^{-1}K$ is closed $\subseteq$ compact so compact. Thus

$F(A \cap F^{-1}K)$ is compact

$= FA \cap K$

Since K is H'ff, $FA \cap K$ is closed in K. $\square$

Cor X space, Y compactly gen'd H'ff, then an embedding $F : X \to Y$ is proper iff $FX \subseteq Y$ closed. $\square$

Cor For $F : X \to Y$ proper cts, Y cgH'ff,

surj $\Rightarrow$ quotient

inj $\Rightarrow$ embedding

bij $\Rightarrow$ homeomorphism. $\square$