17.X.22

Thm For X compact H'ff and $q: X \longrightarrow Y$ a quotient map, TFAE:

(a) Y is H'ff

(b) q is a closed map

(c) $R = \{(x,y) \in X \times X \mid q(x) = q(y)\} \subseteq X \times X$ is closed.

Pf (a) $\Rightarrow$ (b) by CML, (a) $\Rightarrow$ (c) already observed.

(b) $\Rightarrow$ (a): Step 1 Fibers of q are compact: For $y \in Y$ $\exists x \in X$ s.t. $q(x) = y$.

Since $\{x\} \subseteq X$ is closed and q is closed, $q\{x\} = \{y\}$ is closed.

Since q is cts, $q^{-1}\{y\} \subseteq X$ is closed in X compact $\Rightarrow$ compact. ✓

Step 2 Y is H'ff: For $y_1 \neq y_2 \in Y$, we can separate the compact fibers

$q^{-1}\{y_1\}, q^{-1}\{y_2\} \subseteq X$ with disjoint open sets $U_1, U_2 \subseteq X$. Define

$$W_i := \{y \in Y \mid q^{-1}\{y\} \subseteq U_i\}, \quad i = 1, 2.$$

By construction, $y_i \in W_i$ and $W_1 \cap W_2 = \emptyset$. Finally, $W_i = \overbrace{Y \setminus \overbrace{q(\underbrace{X \setminus U_i}_{\text{closed}})}^{\text{closed}}}^{\text{open}}$. ✓

(c) $\Rightarrow$ (a): Assume $\mathcal{R}$ closed. Step 1 Fibers of q are compact:

For $y \in Y$, $x \in X \setminus q^{-1}\{y\}$, let x_1 be a point in $q^{-1}\{y\}$. Since $\mathcal{R}$ closed and $(x_1, x) \in X \times X \setminus \mathcal{R}$, $\exists$ product nbhd $U_1 \times U_2 \subseteq X \times X$ of (x_1, x) disjoint from $\mathcal{R}$. Claim: U_2 is a nbhd of x disjoint from $q^{-1}\{y\}$.

Indeed, $x_2 \in U_2 \cap q^{-1}\{y\} \Rightarrow (x_1, x_2) \in \mathcal{R} \cap (U_1 \times U_2) = \emptyset$. ✓

Thus $X \setminus q^{-1}\{y\}$ open $\Rightarrow q^{-1}\{y\}$ closed $\subseteq$ compact $\Longrightarrow q^{-1}\{y\}$ compact. ✓

Step 2 Y is H'ff: Fix $y_1 \neq y_2 \in Y$. As before, $\exists$ disjoint opens $U_i \supseteq q^{-1}\{y_i\}$, and we can define

$$W_i := \{y \in Y \mid q^{-1}\{y\} \subseteq U_i\}, \quad i = 1, 2.$$

Suffices to show W_i open. Since q is a quotient map, W_i is open iff $q^{-1}W_i$ open iff $X \setminus q^{-1}W_i$ closed. By construction,

$$X \setminus q^{-1}W_i = \{x \in X \mid \exists x' \in X \setminus U_i \text{ s.t. } q(x) = q(x')\}$$
$$= \pi_1(\mathcal{R} \cap (X \times (X \setminus U_i)))$$

proj'n onto first factor

We have π_1 closed by the CML, and $R \cap (X \times (X \setminus U_i))$ is closed by hypothesis. Thus $X \setminus q^{-1} W_i$ is closed. ✓ $\square$

⚠ Closures of coordinate balls might not be homeomorphic to $\overline{\mathbb{B}}^n$.

(E.g. $S^n \setminus \{pt\} \cong \mathbb{B}^n$ with closure S^n

or $\mathbb{R}^n \cong \mathbb{B}^n$ with closure $\mathbb{R}^n$.)

Call a coordinate ball $B \subseteq M$ regular when $\exists$ nbhd B' of $\overline{B}$ and homeo $\varphi : B' \longrightarrow B_{r'}(x) \subseteq \mathbb{R}^n$ taking B to $B_r(x)$ and $\overline{B}$ to $\overline{B}_r(x)$ for some $r' > r > 0$ and $x \in \mathbb{R}^n$.

Lemma Let M be an n-mfld, $B' \subseteq M$ a coordinate ball, $\varphi : B' \longrightarrow B_{r'}(x) \subseteq \mathbb{R}^n$ a homeomorphism. Then $\forall\, 0 < r < r'$, $\varphi^{-1} B_r(x)$ is a regular coordinate ball. $\square$

Prop Every mfld has a countable basis of regular coordinate balls.

Pf Read 4.60. □

Local compactness, paracompactness, & partitions of unity

(1000 ft view)

Locally compact Hausdorff – topological replacement for complete metric spaces

Paracompact – local finiteness condition permitting the development of ...

Partitions of unity – tool for blending locally defined cts maps into a global one.

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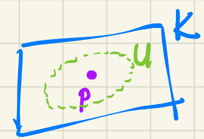
Call X **locally compact** when $\forall p \in X$ $\exists K \subseteq X$ compact containing a nbhd of p.



Call $A \subseteq X$ precompact in X if $\bar{A}$ is compact.

Prop For X H'ff, TFAE

(a) X is locally compact,
(b) Every pt of X has a precompact nbhd,
(c) X has a basis of precompact opens.

Pf (c) $\Rightarrow$ (b) $\Rightarrow$ (a) : ✓

(a) $\Rightarrow$ (c) : Suffices to show that each point $x \in X$ has a nbhd basis of precompact open subsets. (Check: Union of nbhd bases over $x \in X$ is a basis.)

Let $K \subseteq X$ be compact containing a nbhd U of x. Then $\mathcal{V}_x := \{V \subseteq X \mid V$ is a nbhd of x contained in $U\}$ is a nbhd basis of x. WTS all $V \in \mathcal{V}_x$ are precompact. Since X H'ff, K is closed. For $V \in \mathcal{V}$,
$V \subseteq U \subseteq K \Rightarrow \bar{V} \subseteq \bar{K} = K \Rightarrow \bar{V}$ is compact. $\square$

Note Every mfld (w/ or w/o ∂) is locally compact H'ff b/c it has a basis of regular coordinate (half-) balls.

Baire Category Thm Suppose X is locally compact H'ff or a complete metric space. Then every countable collection of dense open subsets of X has a dense intersection.

Pf Reading. $\square$

real polynomials in 2 variables

Application For $f \in \mathbb{R}[x,y]$, $V(f) := \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\}$ is nowhere dense in $\mathbb{R}^2$ (closure has dense complement). So

f nonzero

$U(f) := \mathbb{R}^2 \setminus V(f)$ is dense in $\mathbb{R}^2$. Consider $\mathcal{U}_{\mathbb{Q}} := \{U(f) \mid f \in \mathbb{Q}[x,y]\}$

rational coeffs.

By Baire, $\displaystyle\bigcap_{U(f) \in \mathcal{U}_{\mathbb{Q}}} U(f) \subseteq \mathbb{R}^2$ is dense!

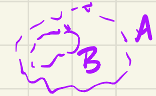
nonzero

I.e. $\exists$ dense set of pts in $\mathbb{R}^2$ satisfying no rational polynomial.

Para compactness ("para" = "alongside" in this case)

- $\mathcal{A} \subseteq 2^X$ is locally finite when $\forall x \in X$ $\exists$ nbhd U of x intersecting finitely many of the sets in $\mathcal{A}$.

- Given a cover $\mathcal{A}$ of X, a cover $\mathcal{B}$ of X is a refinement of $\mathcal{A}$ when $\forall B \in \mathcal{B}$ $\exists A \in \mathcal{A}$ s.t. $B \subseteq A$.



- A space X is paracompact when every open cover of X admits a locally finite open refinement.

Note compact $\subseteq$ paracompact b/c finite subcovers are locally finite open refinements.

Goal Show that mflds are paracompact.

Tool A sequence $(K_i)_{i\in\mathbb{N}}$ of compact subsets of X is an exhaustion of X by compact sets when $X = \bigcup_{i\in\mathbb{N}} K_i$ and $K_i \subseteq K_{i+1}^\circ$ $\forall i \in \mathbb{N}$.

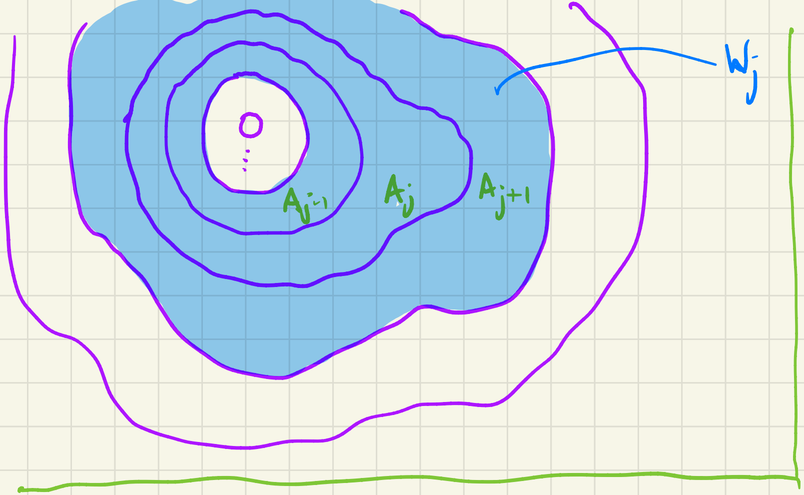
Prop A second countable locally compact H'ff space admits an exhaustion by compact sets.

Pf Take $\{U_i\}_{i\in\mathbb{N}}$ a countable basis of precompact opens. It suffices to construct $(K_j)_{j\in\mathbb{N}}$ with each K_j compact satisfying $U_j \subseteq K_j \subseteq K_{j+1}^\circ$.

Recursive construction: Set $K_0 = \bar{U}_0$. Now assume we have constructed $K_0, \dots, K_n$ that work. Since K_n is compact $\exists\, k_n \in \mathbb{N}$ s.t. $K_n \subseteq U_0 \cup \cdots \cup U_{k_n}$. Define $K_{n+1} := \bar{U}_0 \cup \cdots \cup \bar{U}_{k_n}$. Then K_{n+1} is compact with interior containing K_n. If we also take $k_n > n+1$, then $U_{n+1} \subseteq K_{n+1}$, completing the construction. $\square$

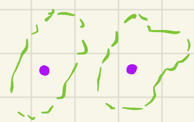
Thm Every 2nd countable locally compact H'ff space (so every mfld w/ or w/o ∂) is paracompact.

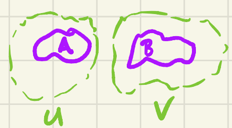
Pf Suppose X is 2nd countable loc cpt H'ff and $\mathcal{U}$ is an open cover of X. Let $(K_j)_{j \in \mathbb{N}}$ be an exhaustion of X by compact sets. For each j, let $A_j := K_{j+1} \setminus K_j^\circ$ and $W_j := K_{j+2}^\circ \setminus K_{j-1}$ (where $K_j = \emptyset$ for $j < 0$). Then $\underbrace{A_j}_{\text{compact}} \subseteq \underbrace{W_j}_{\text{open}}$. For each $x \in A_j$, choose $U_x \in \mathcal{U}$ containing x and set $V_x := U_x \cap W_j$. Then $\{V_x \mid x \in A_j\}$ is an open cover of A_j which



has a finite subcover since A_j is compact. The union of these covers as j ranges through $\mathbb{N}$ is an open cover of X refining $\mathcal{U}$. Since $W_j \cap W_{j'} \neq \emptyset$ only for $j-2 \le j' \le j+2$, this cover is locally finite. $\square$

Normal spaces

Hausdorff: 

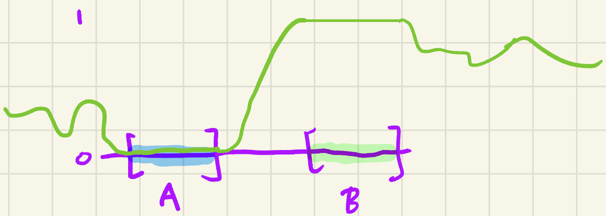
Normal: 

i.e. $\forall A, B \subseteq X$ disjoint closed, $\exists$ disjoint open $U, V \subseteq X$ with $A \subseteq U$, $B \subseteq V$

Thm Every paracompact H'ff space is normal.

Pf Read 4.81. $\square$

Thm (Urysohn's Lemma) Disjoint closed subsets of normal spaces can be separated by cts functions, i.e., if X is normal and $A, B \subseteq X$ are disjoint and closed, then $\exists$ cts $f : X \to [0,1]$ s.t. $A \subseteq f^{-1}\{0\}$, $B \subseteq f^{-1}\{1\}$.



Pavel Urysohn
1898 – 1924

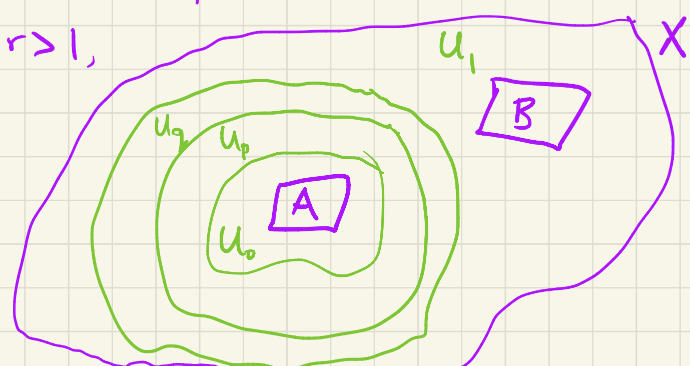
Pf For each $r \in \mathbb{Q}$, we construct $U_r \subseteq X$ open s.t.

(i) $U_r = \emptyset$ for $r < 0$, $U_r = X$ for $r > 1$,

(ii) $U_0 \supseteq A$,

(iii) $U_1 = X \setminus B$, and

(iv) if $p < q$, then $\overline{U_p} \subseteq U_q$



I. Define $U_1 = X \setminus B$ and use (i) to define U_r for $r \notin [0,1] \cap \mathbb{Q}$.
By normality, we may further choose a nbhd U_0 of A s.t. $\bar{U}_0 \subseteq U_1$.
Choose $(r_i)_{i \in \mathbb{N}}$ a sequence enumerating $(0,1) \cap \mathbb{Q}$. By normality, we may choose $U_{r_0} \subseteq X$ s.t. $\bar{U}_0 \subseteq U_{r_0}$, $\bar{U}_{r_0} \subseteq U_1$. For induction, suppose that for $i = 0, \dots, n$ we have open U_{r_i} s.t. $\bar{U}_0 \subseteq U_{r_i}$, $\bar{U}_{r_i} \subseteq U_1$ and $r_i < r_j \Rightarrow \bar{U}_{r_i} \subseteq U_{r_j}$. Define

$$p = \max\{x \in \{0, r_0, \dots, r_n, 1\} \mid x < r_{n+1}\},$$
$$q = \min\{x \in \{0, r_0, \dots, r_n, 1\} \mid x > r_{n+1}\}.$$

By the induction hypothesis, $\bar{U}_p \subseteq U_q$. Now use normality to choose $U_{r_{n+1}} \subseteq X$ open s.t. $\bar{U}_p \subseteq U_{r_{n+1}}$, $\bar{U}_{r_{n+1}} \subseteq U_q$. This completes the inductive construction!

II. Define $f : X \longrightarrow [0,1]$ by $f(x) := \inf\{r \in \mathbb{Q} \mid x \in U_r\}$. By (i), f is well-defined. By (i), (ii), $A \subseteq f^{-1}\{0\}$, and by (i), (iii), $B \subseteq f^{-1}\{1\}$.

Remains to show f is cts; suffices to show $f^{-1}(a,\infty)$, $f^{-1}(-\infty,a)$ open $\forall a \in \mathbb{R}$. Note

$$\circledast \begin{cases} f(x) < a \iff x \in U_r \text{ for some rational } r < a & (\checkmark) \\ f(x) \leq a \iff x \in \overline{U}_r \text{ for all rational } r > a & \text{(need to prove)} \end{cases}$$

Indeed, suppose $f(x) \leq a$. If $r \in \mathbb{Q} \cap (a,\infty)$, then $\exists\, s \in \mathbb{Q} \cap (-\infty, r)$ s.t. $x \in U_s \subseteq U_r \subseteq \overline{U}_r$. For the converse, suppose $x \in \overline{U}_r$ $\forall r \in \mathbb{Q} \cap (a,\infty)$. For $s \in \mathbb{Q} \cap (a,\infty)$, choose $r \in \mathbb{Q} \cap (a,s)$. By hypothesis, $x \in \overline{U}_r \subseteq U_s$, so $f(x) \leq s$. Since this holds $\forall s \in \mathbb{Q} \cap (a,\infty)$, we have $f(x) \leq a$.

By $\circledast$,

$$f^{-1}(-\infty,a) = \bigcup_{r \in \mathbb{Q} \cap (-\infty,a)} U_r$$

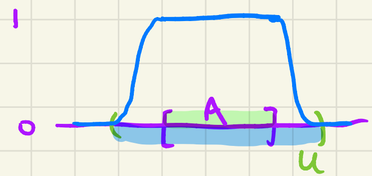
$$f^{-1}(a,\infty) = X \setminus \bigcap_{r \in \mathbb{Q} \cap (a,\infty)} \overline{U}_r$$

which are both open, so f is cts. $\square$

Given $f: X \to \mathbb{R}$ cts, the **support** of f is

$$\operatorname{supp} f := \overline{\{x \in X \mid f(x) \neq 0\}} = \overline{f^{-1}(\mathbb{R} \setminus \{0\})}.$$

For $A \subseteq X$ closed, U a nbhd of A, a cts fn $f: X \to [0,1]$ s.t. $A \subseteq f^{-1}\{1\}$ and $\operatorname{supp} f \subseteq U$ is called a **bump function** for A supported in U.



Cor (bump functions exist) X normal, $A \subseteq X$ closed, U nbhd of A, then $\exists$ a bump fn for A supported in U.

Pf Apply Urysohn's lemma with $B = X \setminus U$. $\square$